MA3151-MATRICES AND CALCULUS

UNIT-3

FUNCTIONS OF SEVERAL VARIABLES

Mr.V.PRAKASH,M.Sc;M.Phil;B.Ed;
Assistant Professor
Department of Mathematics

UNIT - 3

FUNCTIONS OF SEVERVAL VARIABLES

Chapter - 3.1 [Partial Differentiation]

Note:
(i) Let $z = f(x, y)$ be a function of two variable.

then $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ is first order P. D. Equations

(ii) Let $\phi = f(x, y, z)$ be a function of three variable

then $\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}$ & $\frac{\partial \phi}{\partial z}$ is first order P.D. Equation

(iii) Again we differentiate. we get

$$\frac{\partial^2 \phi}{\partial x^2}, \frac{\partial^2 \phi}{\partial y^2}, \frac{\partial^2 \phi}{\partial z^2}, \frac{\partial^2 \phi}{\partial x \partial y}, \frac{\partial^2 \phi}{\partial x \partial z}, \frac{\partial^2 \phi}{\partial y \partial z}.$$

Example - ①

If $z = x + y + xy$, find $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$.

Soln: Given that $z = x + y + xy$

$$\frac{\partial z}{\partial x} = 1 + y$$

$$\frac{\partial z}{\partial y} = 1 + x.$$

Example - ② If $u = \frac{y}{z} + \frac{z}{x}$, find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$.

Soln:

Given that $u = \frac{y}{z} + \frac{z}{x}$.

$$\frac{\partial u}{\partial x} = 0 + z\left(-\frac{1}{x^2}\right) = \frac{-z}{x^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{z} + 0 = \frac{1}{z}$$

$$\frac{\partial u}{\partial z} = y\left(-\frac{1}{z^2}\right) + \frac{1}{x} = \frac{-y}{z^2} + \frac{1}{x}$$

To find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$

$$= x\left[\frac{-z}{x^2}\right] + y\left[\frac{1}{z}\right] + z\left[\frac{-y}{z^2} + \frac{1}{x}\right]$$

$$= \frac{-z}{x} + \frac{y}{z} + \frac{-y\cancel{z}}{z^{\cancel{2}}} + \frac{z}{x}$$

$$= \cancel{\frac{-z}{x}} + \cancel{\frac{y}{z}} - \cancel{\frac{y}{z}} + \cancel{\frac{z}{x}}$$

$$= 0$$

Example - ③ If $u = (x-y)(y-z)(z-x)$, Show that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

Soln:

Given that $u = (x-y)(y-z)(z-x)$

$$u = (x-y)(y-z)(z-x)$$

Here,

$$\frac{\partial u}{\partial x} = (y-z)\left[(x-y)(-1) + (z-x)(1)\right]$$

$$= (y-z)\left[-(x-y) + (z-x)\right]$$

$$= -(y-z)(x-y) + (y-z)(z-x)$$

$$\frac{\partial u}{\partial y} = (z-x)\left[(x-y)(1) + (y-z)(-1)\right]$$

$$= (z-x)\left[(x-y) - (y-z)\right]$$

$$= (z-x)(x-y) - (z-x)(y-z)$$

$$\frac{\partial u}{\partial z} = (x-y)\left[(y-z)(1) + (z-x)(-1)\right]$$

$$= (x-y)\left[(y-z) - (z-x)\right]$$

$$= (x-y)(y-z) - (x-y)(z-x)$$

To Prove $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

L.H.S. $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \cancel{-(y-z)(x-y)} + \cancel{(y-z)(z-x)}$

$$+ \cancel{(z-x)(x-y)} - \cancel{(z-x)(y-z)}$$

$$+ \cancel{(x-y)(y-z)} - \cancel{(x-y)(z-x)}$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

Hence Proved.

Example - (4) If $u = \log(x^2+y^2+z^2)$. Prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{2}{x^2+y^2+z^2}.$$

Soln

Given that $u = \log(x^2+y^2+z^2)$

Diff. p. w. r. to. x,

$$\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2+z^2}(2x)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2+z^2)(2)-(2x)(2x)}{(x^2+y^2+z^2)^2}$$

$$= \frac{2x^2+2y^2+2z^2-4x^2}{(x^2+y^2+z^2)^2}$$

$$= \frac{2y^2+2z^2-2x^2}{(x^2+y^2+z^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{2(y^2+z^2-x^2)}{(x^2+y^2+z^2)^2}$$

|||ly

$$\frac{\partial^2 u}{\partial y^2} = \frac{2(x^2+z^2-y^2)}{(x^2+y^2+z^2)^2}$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{2(x^2+y^2-z^2)}{(x^2+y^2+z^2)^2}$$

To find $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$

$$= \frac{2(y^2+z^2-x^2)}{(x^2+y^2+z^2)^2} + \frac{2(x^2+z^2-y^2)}{(x^2+y^2+z^2)^2} + \frac{2(x^2+y^2-z^2)}{(x^2+y^2+z^2)^2}$$

$$= \frac{2[y^2+z^2-x^2+x^2+z^2-y^2+x^2+y^2-z^2]}{(x^2+y^2+z^2)^2}$$

$$= \frac{2[y^2+z^2-\cancel{x^2}+\cancel{x^2}+\cancel{z^2}-\cancel{y^2}+x^2+\cancel{y^2}-\cancel{z^2}]}{(x^2+y^2+z^2)^2}$$

$$= \frac{2\cancel{[x^2+y^2+z^2]}}{(x^2+y^2+z^2)^{\cancel{2}}}$$

$$= \frac{2}{x^2+y^2+z^2}$$

$$\therefore \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = \frac{2}{x^2+y^2+z^2}.$$

Example-5

If $u=(x^2+y^2+z^2)^{-1/2}$. P.T $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0.$

Soln:

Given that $u=(x^2+y^2+z^2)^{-1/2}$

Diff. p. w.r.to. x.

$$\frac{\partial u}{\partial x} = -\frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-3/2}\cdot(\cancel{2}x)$$

$$= -x(x^2+y^2+z^2)^{-3/2}$$

$$\frac{\partial^2 u}{\partial x^2} = -\left[x\left(-\frac{3}{\cancel{2}}\right)(x^2+y^2+z^2)^{-5/2}(\cancel{2}x)+(x^2+y^2+z^2)^{-3/2}(1)\right]$$

$$= -\left[-3x^2(x^2+y^2+z^2)^{-5/2}+(x^2+y^2+z^2)^{-3/2}\right]$$

$$= \left[3x^2(x^2+y^2+z^2)^{-5/2}-(x^2+y^2+z^2)^{-3/2}\right]$$

$$= (x^2+y^2+z^2)^{-5/2}\left[3x^2-x^2-y^2-z^2\right]$$

$$\frac{\partial^2 u}{\partial x^2} = (x^2+y^2+z^2)^{-5/2}\left[2x^2-y^2-z^2\right] \longrightarrow (1)$$

III ly $\frac{\partial^2 u}{\partial y^2} = (x^2+y^2+z^2)^{-5/2}[2y^2-x^2-z^2] \longrightarrow$ 2

$\frac{\partial^2 u}{\partial z^2} = (x^2+y^2+z^2)^{-5/2}[2z^2-x^2-y^2] \longrightarrow$ 3

Adding 1, 2 & 3, we get

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = (x^2+y^2+z^2)^{-5/2}(2x^2-y^2-z^2)$$
$$+(x^2+y^2+z^2)^{-5/2}(2y^2-x^2-z^2)$$
$$+(x^2+y^2+z^2)^{-5/2}(2z^2-x^2-y^2)$$
$$= [x^2+y^2+z^2]^{-5/2}[\cancel{2x^2}-\cancel{y^2}-\cancel{z^2}+\cancel{2y^2}-\cancel{x^2}-\cancel{z^2}+\cancel{2z^2}-\cancel{x^2}-\cancel{y^2}]$$
$$= [x^2+y^2+z^2]^{-5/2}(0)$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = 0 //$$

Example - 6 If $x = r\cos\theta$ and $y = r\sin\theta$, then find $\frac{\partial r}{\partial x}$ and $\frac{\partial r}{\partial y}$.

Soln Given that $x = r\cos\theta \Rightarrow x^2 = r^2\cos^2\theta$

$y = r\sin\theta \Rightarrow y^2 = r^2\sin^2\theta$

$\therefore x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$= r^2[\cos^2\theta+\sin^2\theta]$

$$\boxed{x^2+y^2 = r^2}$$

Here,
$$r^2 = x^2 + y^2$$
$$r = \sqrt{x^2+y^2}$$

Diff. P. w. r. to. 'x'

$$\frac{\partial r}{\partial x} = \frac{1}{\cancel{2}\sqrt{x^2+y^2}}(\cancel{2}x)$$
$$= \frac{x}{\sqrt{x^2+y^2}}$$
$$\boxed{\frac{\partial r}{\partial x} = \frac{x}{r}}$$

Diff. P. w. r. to. 'y'

$$\frac{\partial r}{\partial y} = \frac{1}{\cancel{2}\sqrt{x^2+y^2}}(\cancel{2}y)$$
$$= \frac{y}{\sqrt{x^2+y^2}}$$
$$\boxed{\frac{\partial r}{\partial y} = \frac{y}{r}}$$

Example-7 If $\phi = f(x-y, y-z, z-x)$, then show that
$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial z} = 0.$$

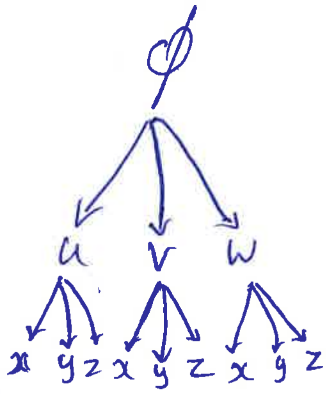
Soln:

Given that $\phi = f(x-y, y-z, z-x)$

ie $\phi = f(u, v, w)$

Here,

$u = x-y$	$v = y-z$	$w = z-x$
$\frac{\partial u}{\partial x} = 1$	$\frac{\partial v}{\partial x} = 0$	$\frac{\partial w}{\partial x} = -1$
$\frac{\partial u}{\partial y} = -1$	$\frac{\partial v}{\partial y} = 1$	$\frac{\partial w}{\partial y} = 0$
$\frac{\partial u}{\partial z} = 0$	$\frac{\partial v}{\partial z} = -1$	$\frac{\partial w}{\partial z} = 1$



$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial \phi}{\partial w}\cdot\frac{\partial w}{\partial x}$$

$$= \frac{\partial \phi}{\partial u}(1) + \frac{\partial \phi}{\partial v}(0) + \frac{\partial \phi}{\partial w}(-1)$$

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial u} - \frac{\partial \phi}{\partial w} \longrightarrow (1)$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v}\cdot\frac{\partial v}{\partial y} + \frac{\partial \phi}{\partial w}\cdot\frac{\partial w}{\partial y}$$

$$= \frac{\partial \phi}{\partial u}(-1) + \frac{\partial \phi}{\partial v}(1) + \frac{\partial \phi}{\partial w}(0)$$

$$\frac{\partial \phi}{\partial y} = -\frac{\partial \phi}{\partial u} + \frac{\partial \phi}{\partial v} \longrightarrow (2)$$

$$\frac{\partial \phi}{\partial z} = \frac{\partial \phi}{\partial u}\cdot\frac{\partial u}{\partial z} + \frac{\partial \phi}{\partial v}\cdot\frac{\partial v}{\partial z} + \frac{\partial \phi}{\partial w}\cdot\frac{\partial w}{\partial z}$$

$$= \frac{\partial \phi}{\partial u}(0) + \frac{\partial \phi}{\partial v}(-1) + \frac{\partial \phi}{\partial w}(1)$$

$$\frac{\partial \phi}{\partial z} = -\frac{\partial \phi}{\partial v} + \frac{\partial \phi}{\partial w} \longrightarrow (3)$$

Adding (1), (2) & (3), we get

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial z} = \cancel{\frac{\partial \phi}{\partial u}} - \cancel{\frac{\partial \phi}{\partial w}} - \cancel{\frac{\partial \phi}{\partial u}} + \cancel{\frac{\partial \phi}{\partial v}} - \cancel{\frac{\partial \phi}{\partial v}} + \cancel{\frac{\partial \phi}{\partial w}}$$

$$= 0. //$$

Hence Proved.

Chapter - 3.2 [Euler's Theorem For Homogeneous Functions]

If u is a homogeneous functions of degree 'n' in x & y, then $\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu}$

Example - (1) Show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u\log u$.

where $\log u = \frac{x^3+y^3}{3x+4y}$.

Soln:- Given that $\log u = \frac{x^3+y^3}{3x+4y}$

Let $z = \log u$

$$= \frac{x^3+y^3}{3x+4y} = \frac{x^{\cancel{3}\,2}\left[1+y^3/x^3\right]}{\cancel{x}\left[3+4y/x\right]}$$

$$z = \frac{x^2\left[1+y^3/x^3\right]}{\left(3+4y/x\right)}$$

$\therefore$ z is a homogeneous function of degree '2', $\boxed{n=2}$

By Euler's theorem $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz \longrightarrow (1)$

Now, $z = \log u$

$$\frac{\partial z}{\partial x} = \frac{1}{u}\cdot\frac{\partial u}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{1}{u}\cdot\frac{\partial u}{\partial y}$$

from ① $\Rightarrow x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz$

$$x\left[\frac{1}{u}\frac{\partial u}{\partial x}\right]+y\left[\frac{1}{u}\cdot\frac{\partial u}{\partial y}\right]=2\log u$$

$$1/u\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right]=2\log u$$

$$\boxed{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2u\log u.}$$

Example - ②

If $u=\tan^{-1}\left[\frac{x^3+y^3}{x-y}\right]$, then Prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u.$

Soln: Given that $u=\tan^{-1}\left[\frac{x^3+y^3}{x-y}\right]\Rightarrow \tan u=\frac{x^3+y^3}{x-y}$

Let $z=\tan u.$

$$=\frac{x^3+y^3}{x-y}=\frac{x^3\left[1+y^3/x^3\right]}{x\left[1-y/x\right]}$$

$$z=\frac{x^2\left[1+y^3/x^3\right]}{\left[1-y/x\right]}$$

$\therefore$ z is a homogenous function of degree $\boxed{n=2}$

By Euler thorem.

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz$$

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2z.\longrightarrow ①$$

Let $z = \tan u$

$$\frac{\partial z}{\partial x} = \sec^2 u \frac{\partial u}{\partial x} \quad \& \quad \frac{\partial z}{\partial y} = \sec^2 u \frac{\partial u}{\partial y}$$

$$(1) \Rightarrow x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z.$$

$$x\left[\sec^2 u \frac{\partial u}{\partial x}\right] + y\left[\sec^2 u \frac{\partial u}{\partial y}\right] = 2\tan u$$

$$\sec^2 u\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] = 2\tan u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\tan u.}{\sec^2 u}$$

$$= 2\frac{\sin u}{\cos u}.\cos^2 u$$

$$= 2\sin u.\cos u$$

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u}$$

Example - (3)

If $u = \cos^{-1}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$. Prove that $x\frac{\partial u}{\partial y} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\cot u$.

Soln

Given that $u = \cos^{-1}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$

$$\cos u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$$

Let $z = \cos u$.

$$Z = \cos u$$

$$= \frac{x+y}{\sqrt{x}+\sqrt{y}} = \frac{x[1+y/x]}{\sqrt{x}[1+\sqrt{y}/\sqrt{x}]}$$

$$= \frac{x[1+y/x]}{x^{1/2}[1+\sqrt{y}/\sqrt{x}]} = \frac{x^{1-1/2}[1+y/x]}{[1+\sqrt{y}/\sqrt{x}]}$$

$$Z = \frac{x^{1/2}[1+y/x]}{[1+\sqrt{y}/\sqrt{x}]}$$

$\therefore$ Z is a homogeneous function of degree $\boxed{n=1/2}$

By Euler's theorem.

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$$

ie, $$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{1}{2}z \longrightarrow (1)$$

Now, $\boxed{z = \cos u}$

$$\frac{\partial z}{\partial x} = -\sin u\frac{\partial u}{\partial x} \quad \& \quad \frac{\partial z}{\partial y} = -\sin u\frac{\partial u}{\partial y}$$

$(1) \Rightarrow$ $$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{1}{2}z.$$

$$x\left[-\sin u\frac{\partial u}{\partial x}\right] + y\left[-\sin u\frac{\partial u}{\partial x}\right] = \frac{1}{2}\cos u$$

$$-\sin u\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] = \frac{1}{2}\cos u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\frac{\cos u}{-\sin u}$$

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\cot u.}$$

Example - 4 :

If $z = \log(x^2+xy+y^2)$. Show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2$, using Euler's theorem.

Soln.

Given that $z = \log(x^2+xy+y^2)$

$$e^z = (x^2+xy+y^2)$$

Let $f = e^z = (x^2+xy+y^2)$

$$= x^2\left[1+y/x+y^2/x^2\right]$$

$\therefore$ f is homogeneous function of degree $\boxed{n=2}$

By Euler's theorem,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

ie, $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 2f \longrightarrow (1)$

Let $f = e^z$

$$\frac{\partial f}{\partial x} = e^z\frac{\partial z}{\partial x} \quad \frac{\partial f}{\partial y} = e^z\frac{\partial z}{\partial y}.$$

$(1) \Rightarrow x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 2e^z$

$$x\left[e^z\frac{\partial z}{\partial x}\right] + y\left[e^z\frac{\partial z}{\partial y}\right] = 2e^z$$

$$e^z\left[x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right] = 2e^z$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2\frac{\cancel{e^z}}{\cancel{e^z}} = 2$$

Example - 5

Verify Euler's theorem for the function $u = x^2+y^2+2xy$.

Soln:-

Given that $u = x^2+y^2+2xy$

It is clear that u is a homogeneous function of degree $\boxed{n=2}$

By Euler's theorem,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = nu$$

i.e., $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = 2u \longrightarrow (1)$$

Let $u = x^2+y^2+2xy$

$$\frac{\partial u}{\partial x} = 2x+2y \quad \& \quad \frac{\partial u}{\partial y} = 2y+2x$$

$$(1) \Rightarrow \quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = 2u$$

$$x(2x+2y)+y(2y+2x) = 2(x^2+y^2+2xy)$$

$$2x^2+2xy+2y^2+2xy = 2(x^2+y^2+2xy)$$

$$2x^2+2y^2+4xy = 2(x^2+y^2+2xy)$$

$$2(x^2+y^2+2xy) = 2(x^2+y^2+2xy)$$

$\therefore$ Hence Euler's theorem is verified.

Example - 6

If $u = \sin^{-1}\left(\frac{x^3 - y^3}{x+y}\right)$. Prove that

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\tan u$ by using Euler's theorem.

Soln.

Given that $u = \sin^{-1}\left[\frac{x^3 - y^3}{x+y}\right]$

$$\sin u = \frac{x^3 - y^3}{x+y}$$

Let $z = \sin u$

$$= \frac{x^3 - y^3}{x+y} = \frac{x^{3}\left[1 - y^3/x^3\right]}{x\left[1 + y/x\right]}$$

$$z = \frac{x^2\left[1 - y^3/x^3\right]}{\left[1 + y/x\right]}$$

$\therefore$ z is a homogeneous function of degree $\boxed{n=2}$

By Euler's Theorem.

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$$

ie, $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z \longrightarrow (1)$

Let $z = \sin u$

$$\frac{\partial z}{\partial x} = \cos u \cdot \frac{\partial u}{\partial x} \quad \& \quad \frac{\partial z}{\partial y} = \cos u \cdot \frac{\partial u}{\partial y}$$

$(1) \Rightarrow x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$

$$x\left[\cos u \cdot \frac{\partial u}{\partial x}\right] + y\left[\cos u \frac{\partial u}{\partial y}\right] = 2\sin u$$

$$\cos u\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] = 2\sin u$$

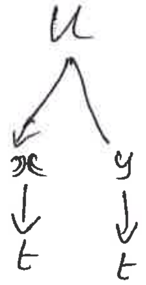
$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\sin u}{\cos u} = 2\tan u$$

Chapter-3.3 [Total Differential Coefficient]

If $u = f(x, y)$ is a function of x & y.

and $x = f(t)$ & $y = f(t)$. then find $\frac{du}{dt} = ?$

$$\therefore \frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}.$$



Example-①

Find $\frac{du}{dt}$ if $u = x^3 y^4$, where $x = t^3$, $y = t^2$.

Solun

Given that $u = x^3 y^4$; $x = t^3$; $y = t^2$

$$\frac{\partial u}{\partial x} = 3x^2 y^4 \quad \Big| \quad \frac{dx}{dt} = 3t^2 \quad \Big| \quad \frac{dy}{dt} = 2t.$$

$$\frac{\partial u}{\partial y} = 4x^3 y^3$$

W.K.T
$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

$$= 3x^2 y^4 (3t^2) + 4x^3 y^3 (2t)$$

$$= 3(t^3)^2 (t^2)^4 (3t^2) + 4(t^3)^3 (t^2)^3 (2t)$$

$$= 9(t)^6 (t)^8 (t)^2 + 8(t)^9 (t)^6 (t)$$

$$= 9t^{16} + 8t^{16}$$

$$\frac{du}{dt} = 17t^{16} //$$

Example - (2) Find $\frac{du}{dx}$, if $u = \tan^{-1}(x/y)$, where $x^2+y^2=a^2$.

Soln. The total differential of u is

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$\div$ by dx on b.s

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\frac{dx}{dx} + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} \longrightarrow (1)$$

Given that $u = \tan^{-1}(x/y)$

$$\frac{\partial u}{\partial x} = \frac{1}{1+x^2/y^2}(1/y) = \frac{1}{\frac{y^2+x^2}{\cancel{y^2}}}(1/\cancel{y})$$

$$\frac{\partial u}{\partial x} = \frac{y}{x^2+y^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{1+x^2/y^2}(-x/y^2) = \frac{1}{\frac{y^2+x^2}{\cancel{y^2}}}\cdot(-x/\cancel{y^2})$$

$$\frac{\partial u}{\partial y} = \frac{-x}{x^2+y^2}$$

Also, $x^2+y^2=a^2$

Diff. w.r.to. 'x', we get $2x + 2y\frac{dy}{dx} = 0$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-\cancel{2}x}{\cancel{2}y} = -x/y$$

$$\boxed{\frac{dy}{dx} = -x/y}$$

$$① \Longrightarrow \frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

$$= \frac{y}{x^2+y^2} + \frac{-x}{x^2+y^2}\left(\frac{-x}{y}\right)$$

$$= \frac{y}{x^2+y^2} + \frac{(x^2/y)}{x^2+y^2}$$

$$= \frac{y + x^2/y}{x^2+y^2} = \frac{\left(\frac{y^2+x^2}{y}\right)}{x^2+y^2}$$

$$= \frac{\left(x^2+y^2/y\right)}{x^2+y^2} = \frac{\cancel{(x^2+y^2)}}{y\cancel{(x^2+y^2)}}$$

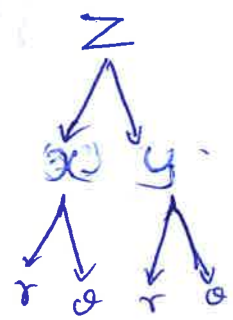
$$\boxed{\frac{du}{dx} = 1/y}$$

Example-③ If $z = f(x,y)$ where $x = r\cos\theta$; $y = r\sin\theta$

Show that $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2.$

Soln:

Given that $z = f(x,y)$ and $x = r\cos\theta$; $y = r\sin\theta$



let $x = r\cos\theta$	$y = r\sin\theta$
$\frac{\partial x}{\partial r} = \cos\theta$	$\frac{\partial y}{\partial r} = \sin\theta$
$\frac{\partial x}{\partial \theta} = r(-\sin\theta)$	$\frac{\partial y}{\partial \theta} = r\cos\theta$
$= -r\sin\theta$	

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}(\cos\theta) + \frac{\partial z}{\partial y}(\sin\theta) \longrightarrow (1)$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}$$

$$= \frac{\partial z}{\partial x}(-r\sin\theta) + \frac{\partial z}{\partial y}(r\cos\theta)$$

$$= r\left[-\frac{\partial z}{\partial x}\sin\theta + \frac{\partial z}{\partial y}\cos\theta\right]$$

$$\frac{1}{r}\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial y}\cos\theta - \frac{\partial z}{\partial x}\sin\theta \longrightarrow (2)$$

Squaring w.r.t

$$(1)^2 \Rightarrow \left(\frac{\partial z}{\partial r}\right)^2 = \left[\frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta\right]^2$$

$$= \left(\frac{\partial z}{\partial x}\right)^2\cos^2\theta + \left(\frac{\partial z}{\partial y}\right)^2\sin^2\theta + 2\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)\cos\theta\sin\theta \rightarrow (3)$$

$$(2)^2 \Rightarrow \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left[\frac{\partial z}{\partial y}\cos\theta - \frac{\partial z}{\partial x}\sin\theta\right]^2$$

$$= \left(\frac{\partial z}{\partial y}\right)^2\cos^2\theta + \left(\frac{\partial z}{\partial x}\right)^2\sin^2\theta - 2\left(\frac{\partial z}{\partial y}\right)\left(\frac{\partial z}{\partial x}\right)\cos\theta\sin\theta \rightarrow (4)$$

Adding ③ & ④

$$③ + ④ \Rightarrow \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 \cos^2\theta + \left(\frac{\partial z}{\partial y}\right)^2 \sin^2\theta$$

$$+ \cancel{2\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)\cos\theta \sin\theta} + \left(\frac{\partial z}{\partial y}\right)^2 \cos^2\theta$$

$$+ \left(\frac{\partial z}{\partial x}\right)^2 \sin^2\theta - \cancel{2\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right)\cos\theta \sin\theta}$$

$$= \left(\frac{\partial z}{\partial x}\right)^2 \cos^2\theta + \left(\frac{\partial z}{\partial y}\right)^2 \sin^2\theta + \left(\frac{\partial z}{\partial y}\right)^2 \cos^2\theta$$

$$+ \left(\frac{\partial z}{\partial x}\right)^2 \sin^2\theta$$

$$= \left(\frac{\partial z}{\partial x}\right)^2 \left[\cos^2\theta + \sin^2\theta\right] + \left(\frac{\partial z}{\partial y}\right)^2 \left[\cos^2\theta + \sin^2\theta\right]$$

$$= \left(\frac{\partial z}{\partial x}\right)^2 (1) + \left(\frac{\partial z}{\partial y}\right)^2 (1)$$

$$\therefore \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

Example - ④

If $\phi = f(u, v)$, $u = e^x \cos y$, $v = e^x \sin y$.

Show that $$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = (u^2 + v^2)\left[\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2}\right].$$

Soln:

Given that $u = e^x \cos y$ | $v = e^x \sin y$

$$\frac{\partial u}{\partial x} = e^x \cos y \qquad \frac{\partial v}{\partial x} = e^x \sin y$$

$$\frac{\partial u}{\partial y} = e^x(-\sin y) \qquad \frac{\partial v}{\partial y} = e^x \cos y.$$

$$= -e^x \sin y$$

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$= \frac{\partial \phi}{\partial u}(e^x \cos y) + \frac{\partial \phi}{\partial v}(e^x \sin y)$$

$$\frac{\partial \phi}{\partial x} = u\frac{\partial \phi}{\partial u} + v\frac{\partial \phi}{\partial v} \longrightarrow (1)$$

$$\partial/\partial x = u\,\partial/\partial u + v\,\partial/\partial v \longrightarrow (2)$$

$$\therefore \frac{\partial^2 \phi}{\partial x^2} = \partial/\partial x\left(\frac{\partial \phi}{\partial x}\right) = \left(u\,\partial/\partial u + v\,\partial/\partial v\right)\left(u\frac{\partial \phi}{\partial u} + v\frac{\partial \phi}{\partial v}\right)$$

$$\frac{\partial^2 \phi}{\partial x^2} = u^2\frac{\partial^2 \phi}{\partial u^2} + uv\frac{\partial^2 \phi}{\partial u \partial v} + uv\frac{\partial^2 \phi}{\partial u \partial v} + v^2\frac{\partial \phi}{\partial v^2} \longrightarrow (3)$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$= \frac{\partial \phi}{\partial u}(-e^x \sin y) + \frac{\partial \phi}{\partial v}(e^x \cos y)$$

$$= \frac{\partial \phi}{\partial u}(-v) + \frac{\partial \phi}{\partial v}(u)$$

$$\frac{\partial \phi}{\partial y} = u\frac{\partial \phi}{\partial v} - v\frac{\partial \phi}{\partial u} \longrightarrow (4)$$

$$\partial/\partial y = u\,\partial/\partial v - v\,\partial/\partial u \longrightarrow (5)$$

$$\frac{\partial^2 \phi}{\partial y^2} = \partial/\partial x(\partial \phi/\partial y) = \left(u\,\partial/\partial v - v\,\partial/\partial u\right)\left(u\frac{\partial \phi}{\partial v} - v\,\partial \phi/\partial u\right)$$

$$\frac{\partial^2 \phi}{\partial y^2} = u^2\frac{\partial^2 \phi}{\partial v^2} - uv\frac{\partial^2 \phi}{\partial u \partial v} - uv\frac{\partial^2 \phi}{\partial u \partial v} + v^2\frac{\partial^2 \phi}{\partial u^2} \longrightarrow (6)$$

Adding (3) & (6), we get

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=u^2\frac{\partial^2\phi}{\partial u^2}+\cancel{2uv\frac{\partial^2\phi}{\partial u\partial v}}+v^2\frac{\partial^2\phi}{\partial v^2}+u^2\frac{\partial^2\phi}{\partial v^2}$$
$$-\cancel{2uv\frac{\partial^2\phi}{\partial u\partial v}}+v^2\frac{\partial^2\phi}{\partial u^2}$$
$$=u^2\frac{\partial^2\phi}{\partial u^2}+v^2\frac{\partial^2\phi}{\partial v^2}+u^2\frac{\partial^2\phi}{\partial v^2}+v^2\frac{\partial^2\phi}{\partial u^2}$$
$$=u^2\left[\frac{\partial^2\phi}{\partial u^2}+\frac{\partial^2\phi}{\partial v^2}\right]+v^2\left[\frac{\partial^2\phi}{\partial u^2}+\frac{\partial^2\phi}{\partial v^2}\right]$$
$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=(u^2+v^2)\left[\frac{\partial^2\phi}{\partial u^2}+\frac{\partial^2\phi}{\partial v^2}\right]$$ //

Example-5

If $u=f(r,s,t)$, $r=\frac{x}{y}$, $s=\frac{y}{z}$, $t=\frac{z}{x}$.

Show that $x\frac{\partial u}{\partial x}+y\frac{\partial v}{\partial y}+z\frac{\partial u}{\partial z}=0$.

soln:

Given that $r=\frac{x}{y}$ | $s=\frac{y}{z}$ | $t=\frac{z}{x}$

$\frac{\partial r}{\partial x}=\frac{1}{y}$ | $\frac{\partial s}{\partial x}=0$ | $\frac{\partial t}{\partial x}=-\frac{z}{x^2}$

$\frac{\partial r}{\partial y}=\frac{-x}{y^2}$ | $\frac{\partial s}{\partial y}=\frac{1}{z}$ | $\frac{\partial t}{\partial y}=0$

$\frac{\partial r}{\partial z}=0$ | $\frac{\partial s}{\partial z}=-\frac{y}{z^2}$ | $\frac{\partial t}{\partial z}=\frac{1}{x}$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x}+\frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}$$
$$=\frac{\partial u}{\partial r}\left(\frac{1}{y}\right)+\frac{\partial u}{\partial s}(0)+\frac{\partial u}{\partial t}\cdot\left(-\frac{z}{x^2}\right)$$
$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\left(\frac{1}{y}\right)-\frac{\partial u}{\partial t}\left(\frac{z}{x^2}\right)\longrightarrow(1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}$$

$$= \frac{\partial u}{\partial r}\left(\frac{-x}{y^2}\right) + \frac{\partial u}{\partial s}\left(\frac{1}{z}\right) + \frac{\partial u}{\partial t}(0)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial u}{\partial r}\left(\frac{x}{y^2}\right) + \frac{\partial u}{\partial s}\left(\frac{1}{z}\right) \longrightarrow 2$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z}$$

$$= \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}\left(-\frac{y}{z^2}\right) + \frac{\partial u}{\partial t}\left(\frac{1}{x}\right)$$

$$\frac{\partial u}{\partial z} = -\frac{\partial u}{\partial s}\left(\frac{y}{z^2}\right) + \frac{\partial u}{\partial t}\left(\frac{1}{x}\right) \longrightarrow 3$$

<u>To find</u>: $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$

$$= x\left[\frac{\partial u}{\partial r}\left(\frac{1}{y}\right) - \frac{\partial u}{\partial t}\left(\frac{z}{x^2}\right)\right] + y\left[-\frac{\partial u}{\partial r}\left(\frac{x}{y^2}\right) + \frac{\partial u}{\partial s}\left(\frac{1}{z}\right)\right]$$
$$+ z\left[-\frac{\partial u}{\partial s}\left(\frac{y}{z^2}\right) + \frac{\partial u}{\partial t}\left(\frac{1}{x}\right)\right]$$

$$= \cancel{\frac{\partial u}{\partial r}\left(\frac{x}{y}\right)} - \cancel{\frac{\partial u}{\partial t}\left(\frac{z}{x}\right)} - \cancel{\frac{\partial u}{\partial r}\left(\frac{x}{y}\right)} + \cancel{\frac{\partial u}{\partial s}\left(\frac{y}{z}\right)} - \cancel{\frac{\partial u}{\partial s}\left(\frac{y}{z}\right)}$$
$$+ \cancel{\frac{\partial u}{\partial t}\left(\frac{z}{x}\right)}$$

$$= 0.$$

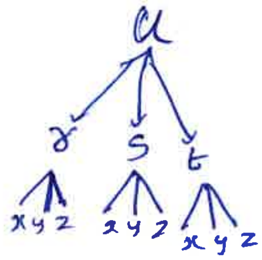
$$\therefore\ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$$

Example - 6

If $u = f(2x-3y,\ 3y-4z,\ 4z-2x)$, then find

$$\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z}.$$

Soln:



Given $u = f(2x-3y,\ 3y-4z,\ 4z-2x)$

i.e. $u = f(r, s, t)$

Here, $r = 2x-3y$ | $s = 3y-4z$ | $t = 4z-2x$

$\frac{\partial r}{\partial x} = 2$ | $\frac{\partial s}{\partial x} = 0$ | $\frac{\partial t}{\partial x} = -2$

$\frac{\partial r}{\partial y} = -3$ | $\frac{\partial s}{\partial y} = 3$ | $\frac{\partial t}{\partial y} = 0$

$\frac{\partial r}{\partial z} = 0$ | $\frac{\partial s}{\partial z} = -4$ | $\frac{\partial t}{\partial z} = 4$

Now,
$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$= \frac{\partial u}{\partial r}(2) + \frac{\partial u}{\partial s}(0) + \frac{\partial u}{\partial t}(-2)$$

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial r} - 2\frac{\partial u}{\partial t} \longrightarrow (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}$$

$$= \frac{\partial u}{\partial r}(-3) + \frac{\partial u}{\partial s}(3) + \frac{\partial u}{\partial t}(0)$$

$$\frac{\partial u}{\partial y} = -3\frac{\partial u}{\partial r} + 3\frac{\partial u}{\partial s} \longrightarrow (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial z}$$

$$= \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}(-4) + \frac{\partial u}{\partial t}(4)$$

$$\frac{\partial u}{\partial z} = -4\frac{\partial u}{\partial s} + 4\frac{\partial u}{\partial t} \longrightarrow (3)$$

To find: $\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z}$

$$= \frac{1}{2}\left[2\frac{\partial u}{\partial r} - 2\frac{\partial u}{\partial t}\right] + \frac{1}{3}\left[-3\frac{\partial u}{\partial r} + 3\frac{\partial u}{\partial s}\right]$$

$$+ \frac{1}{4}\left[-4\frac{\partial u}{\partial s} + 4\frac{\partial u}{\partial t}\right]$$

$$= \cancel{\frac{2}{2}}\left[\frac{\partial u}{\partial r} - \frac{\partial u}{\partial t}\right] + \cancel{\frac{3}{3}}\left[-\frac{\partial u}{\partial r} + \frac{\partial u}{\partial s}\right] + \cancel{\frac{4}{4}}\left[-\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t}\right]$$

$$= \cancel{\frac{\partial u}{\partial r}} - \cancel{\frac{\partial u}{\partial t}} - \cancel{\frac{\partial u}{\partial r}} + \cancel{\frac{\partial u}{\partial s}} - \cancel{\frac{\partial u}{\partial s}} + \cancel{\frac{\partial u}{\partial t}}$$

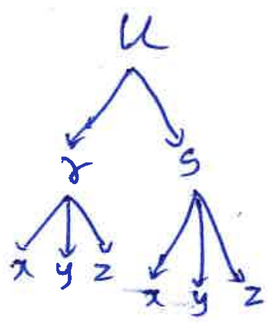
$$= 0.$$

$$\therefore \frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = 0 //$$

Example-7 If $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$, find the value of $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z}$.

Soln: Given that $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$

is $u = f(r, s)$



Here, $r = \dfrac{y-x}{xy}$

$$r = \frac{y}{xy} - \frac{x}{xy}$$

$$r = 1/x - 1/y$$

$$\frac{\partial r}{\partial x} = -1/x^2$$

$$\frac{\partial r}{\partial y} = 1/y^2$$

$$\frac{\partial r}{\partial z} = 0$$

$$s = \frac{z-x}{xz}$$

$$= \frac{z}{xz} - \frac{x}{xz}$$

$$s = 1/x - 1/z$$

$$\frac{\partial s}{\partial x} = -1/x^2$$

$$\frac{\partial s}{\partial y} = 0$$

$$\frac{\partial s}{\partial z} = 1/z^2$$

Now, $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial r}\dfrac{\partial r}{\partial x} + \dfrac{\partial u}{\partial s}\cdot\dfrac{\partial s}{\partial x}$

$$= \frac{\partial u}{\partial r}(-1/x^2) + \frac{\partial u}{\partial s}(-1/x^2)$$

$$\frac{\partial u}{\partial x} = (-1/x^2)\frac{\partial u}{\partial r} - (1/x^2)\frac{\partial u}{\partial r} \longrightarrow (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y}$$

$$= \frac{\partial u}{\partial r}(1/y^2) + \frac{\partial u}{\partial s}(0)$$

$$\frac{\partial u}{\partial y} = (1/y^2)\frac{\partial u}{\partial r} \longrightarrow (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\left(\frac{\partial r}{\partial z}\right) + \frac{\partial u}{\partial s}\frac{\partial s}{\partial z}$$

$$= \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}(1/z^2)$$

$$\frac{\partial u}{\partial z} = (1/z^2)\frac{\partial u}{\partial s} \longrightarrow (3)$$

To find: $x^2\frac{\partial u}{\partial x}+y^2\frac{\partial u}{\partial y}+z^2\frac{\partial u}{\partial z}$

$$=x^2\left[-\tfrac{1}{x^2}\frac{\partial u}{\partial r}-\tfrac{1}{x^2}\frac{\partial u}{\partial s}\right]+y^2\left(\tfrac{1}{y^2}\right)\frac{\partial u}{\partial r}+z^2\left(\tfrac{1}{z^2}\right)\frac{\partial u}{\partial s}$$

$$=\frac{\cancel{x^2}}{\cancel{x^2}}\left[-\frac{\partial u}{\partial r}-\frac{\partial u}{\partial s}\right]+\frac{\cancel{y^2}}{\cancel{y^2}}\left(\frac{\partial u}{\partial r}\right)+\frac{\cancel{z^2}}{\cancel{z^2}}\left(\frac{\partial u}{\partial s}\right)$$

$$=\cancel{-\frac{\partial u}{\partial r}}\cancel{-\frac{\partial u}{\partial s}}+\cancel{\frac{\partial u}{\partial r}}+\cancel{\frac{\partial u}{\partial s}}$$

$$=0.$$

$$\therefore\ x^2\frac{\partial u}{\partial x}+y^2\frac{\partial u}{\partial y}+z^2\frac{\partial u}{\partial z}=0.$$

Chapter-3.4 [Jacobians]

Note:

(i) $$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

(ii) $$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Example-①

If $x = r\cos\theta$; $y = r\sin\theta$, then find $\frac{\partial(x,y)}{\partial(r,\theta)}$

Soln

Given that $x = r\cos\theta$ | $y = r\sin\theta$

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial y}{\partial r} = \sin\theta$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$$

W.K.T $$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r[\cos^2\theta + \sin^2\theta]$$

$$= r(1)$$

$$= r //$$

Example-② If $x = uv$, $y = \frac{u}{v}$, then find $\frac{\partial(x,y)}{\partial(u,v)}$.

Soln:-

Given that $x = uv$ | $y = u/v$

$\frac{\partial x}{\partial u} = v$ | $\frac{\partial y}{\partial u} = 1/v$

$\frac{\partial x}{\partial v} = u$ | $\frac{\partial y}{\partial v} = -u/v^2$

W.K.T $J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 1/v & -u/v^2 \end{vmatrix}$

$= \cancel{v}\left(\frac{-u}{v^{\cancel{2}}}\right) - u\left(1/v\right)$

$= \left(\frac{-u}{v}\right) - u/v$

$\frac{\partial(x,y)}{\partial(u,v)} = -\frac{2u}{v}$ //

Example-③ If $x = u(1+v)$, $y = v(1+u)$, find $\frac{\partial(x,y)}{\partial(u,v)}$.

Soln.

Given that $x = u(1+v)$ | $y = v(1+u)$

$x = u + uv$ | $y = v + uv$

$\frac{\partial x}{\partial u} = 1+v$ | $\frac{\partial y}{\partial u} = v$

$\frac{\partial x}{\partial v} = u$ | $\frac{\partial y}{\partial v} = 1+u$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix} = (1+v)(1+u) - uv$$

$$= 1+u+v+\cancel{uv}-\cancel{uv}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = 1+u+v \;//$$

Example - (4)

If $x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$

then find $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = ?$

Soln:

Given $x = r\sin\theta\cos\phi$ | $y = r\sin\theta\sin\phi$ | $z = r\cos\theta$

$\frac{\partial x}{\partial r} = \sin\theta\cos\phi$ | $\frac{\partial y}{\partial r} = \sin\theta\sin\phi$ | $\frac{\partial z}{\partial r} = \cos\theta$

$\frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi$ | $\frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi$ | $\frac{\partial z}{\partial \theta} = -r\sin\theta$

$\frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi$ | $\frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi$ | $\frac{\partial z}{\partial \phi} = 0$

To find:

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \sin\theta\cos\phi\left[0 + r^2\sin^2\theta\cos\phi\right] - r\cos\theta\cos\phi\left[0 - r\sin\theta\cos\theta\cos\phi\right]$$
$$- r\sin\theta\sin\phi\left[-r\sin^2\theta\sin\phi - r\cos^2\theta\sin\phi\right]$$

$$= r^2\sin^3\theta\cos^2\phi + r^2\sin\theta\cos^2\theta\cos^2\phi + r^2\sin^3\theta\sin^2\phi$$
$$+ r^2\sin\theta\cos^2\theta\sin^2\phi$$

$$= r^2\sin^3\theta\left[\cos^2\phi + \sin^2\phi\right] + r^2\sin\theta\cos^2\theta\left[\cos^2\phi + \sin^2\phi\right]$$

$$= r^2\sin^3\theta(1) + r^2\sin\theta\cos^2\theta(1)$$

$$= r^2\sin^3\theta + r^2\sin\theta\cos^2\theta$$

$$= r^2\sin\theta\left[\sin^2\theta + \cos^2\theta\right]$$

$$= r^2\sin\theta[1]$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta \;//$$

Example-5

If $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$, find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

Soln:

Given

$$u = \frac{yz}{x} \quad \Big| \quad v = \frac{zx}{y} \quad \Big| \quad w = \frac{xy}{z}$$

$$u_x = \frac{-yz}{x^2} \quad \Big| \quad v_x = \frac{z}{y} \quad \Big| \quad w_x = \frac{y}{z}$$

$$u_y = \frac{z}{x} \quad \Big| \quad v_y = \frac{-zx}{y^2} \quad \Big| \quad w_y = \frac{x}{z}$$

$$u_z = \frac{y}{x} \quad \Big| \quad v_z = \frac{x}{y} \quad \Big| \quad w_z = \frac{-xy}{z^2}$$

To find:- $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$

$$= \begin{vmatrix} \frac{-yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & \frac{-zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & \frac{-xy}{z^2} \end{vmatrix}$$

$$= \frac{1}{x^2y^2z^2} \begin{vmatrix} -yz & xz & xy \\ yz & -zx & xy \\ yz & xz & -xy \end{vmatrix}$$

$$= \frac{x^2y^2z^2}{x^2y^2z^2} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= (1)\left[-1(1-1) - 1(-1-1) + 1(1+1)\right]$$

$$= \left[0 - 1(-2) + 1(2)\right] = 2+2 = 4$$

$$J = 4$$

Example-6 If $u = x+y+z$, $v = xy+yz+zx$, $w = x^2+y^2+z^2$

then find $\frac{\partial(u,v,w)}{\partial(x,y,z)} = ?$

Soln:

Given

$u = x+y+z$	$v = xy+yz+zx$	$w = x^2+y^2+z^2$
$u_x = 1$	$v_x = y+z$	$w_x = 2x$
$u_y = 1$	$v_y = x+z$	$w_y = 2y$
$u_z = 1$	$v_z = y+x$	$w_z = 2z$

To find:

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ y+z & x+z & y+x \\ 2x & 2y & 2z \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ y+z & x+z & x+y \\ x & y & z \end{vmatrix}$$

$$\cancel{= 2[1(x+z-z) - 1(y+z}$$

$$= 2\{1[(x+z)z - y(x+y)] - 1[z(y+z) - x(x+y)] + 1[y(y+z) - x(x+z)]\}$$

$$= 2\{\cancel{xz} + \cancel{z^2} - \cancel{xy} - \cancel{y^2} - \cancel{yz} - \cancel{z^2} + x^2 + \cancel{xy} + \cancel{y^2} + \cancel{yz} - x^2 - \cancel{xz}\}$$

$$= 2(0)$$

$$J = 2$$

Chapter-3.5 [Taylor's Theorem for Functions of two variables]

note:

$$f(x,y) = f(a,b) + [h f_x(a,b) + k f_y(a,b)]$$
$$+ \tfrac{1}{2}[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)]$$
$$+ \tfrac{1}{6}[h^3 f_{xxx}(a,b) + 3h^2k f_{xxy}(a,b) + 3hk^2 f_{xyy}(a,b) + k^3 f_{yyy}(a,b)]$$
$$+ \cdots\cdots$$

where $\boxed{h = x-a}$ & $\boxed{k = y-b}$.

Example-①

Expand $e^x \sin y$ in powers of x and y upto third degree terms by using Taylor's series.

Soln:

Given that $f(x,y) = e^x \sin y$, Here $a=0$, $b=0$

and $h = x-a$ & $k = y-b$

$h = x-0$ & $k = y-0$

$\boxed{h = x}$ & $\boxed{k = y}$

W-K-T

$\boxed{e^0 = 1}$; $\boxed{\sin(0) = 0}$; $\boxed{\cos(0) = 1}$

$f(x,y) = e^x \sin y$	$f(0,0) = e^0 \sin(0) = 0$
$f_x(x,y) = e^x \sin y$ $f_y(x,y) = e^x \cos y$	$f_x(0,0) = e^0 \sin(0) = 0$ $f_y(0,0) = e^0 \cos(0) = 1$
$f_{xx}(x,y) = e^x \sin y$ $f_{xy}(x,y) = e^x \cos y$ $f_{yy}(x,y) = -e^x \sin y$	$f_{xx}(0,0) = e^0 \sin(0) = 0$ $f_{xy}(0,0) = e^0 \cos(0) = 1$ $f_{yy}(0,0) = -e^0 \sin(0) = 0$
$f_{xxx}(x,y) = e^x \sin y$ $f_{xxy}(x,y) = e^x \cos y$ $f_{yyx}(x,y) = -e^x \sin y$ $f_{yyy}(x,y) = -e^x \cos y$	$f_{xxx}(0,0) = e^0 \sin(0) = 0$ $f_{xxy}(0,0) = e^0 \cos(0) = 1$ $f_{yyx}(0,0) = -e^0 \sin(0) = 0$ $f_{yyy}(0,0) = -e^0 \cos(0) = -1$

Taylor's theorem,

$$f(x,y) = f(a,b) + [h f_x(a,b) + k f_y(a,b)]$$
$$+ \tfrac{1}{2}[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)]$$
$$+ \tfrac{1}{6}[h^3 f_{xxx}(a,b) + 3h^2k f_{xxy}(a,b) + 3hk^2 f_{xyy}(a,b) + k^3 f_{yyy}] + \cdots$$

$$= (0) + [x(0) + y(1)] + \tfrac{1}{2}[x^2(0) + 2xy(1) + y^2(0)]$$
$$+ \tfrac{1}{6}[x^3(0) + 3x^2y(1) + 3xy^2(0) + y^3(-1)]$$

$$f(x,y) = 0 + [0 + y] + \tfrac{1}{2}\left[0 + 2xy + \cancel{0}\right] + \tfrac{1}{6}\left[0 + 3x^2y + 0 - y^3\right]$$

$$= y + \tfrac{1}{2}[2xy + \cancel{0}] + \tfrac{1}{6}[3x^2y - y^3]$$

$$= y + xy + \frac{\cancel{3}x^2y}{\cancel{6}_2} - \frac{y^3}{6}$$

$$\therefore e^x \sin y = y + xy + \frac{x^2y}{2} - \frac{y^3}{6}.$$

Example - ②

Expand $e^x \cos y$ about $(0, \pi/2)$ upto third degree using Taylor's series.

Soln:

Given that $f(x,y) = e^x \cos y$ and $\boxed{a=0}$ $\boxed{b=\pi/2}$

Here, $h = x - a$ | $k = y - b$

$= x - 0$ | $= y - \pi/2$

$\boxed{h = x}$ | $\boxed{k = y - \pi/2}$

W.K.T $\boxed{e^0 = 1}$ $\boxed{\cos \pi/2 = 0}$ $\boxed{\sin \pi/2 = 1}$

$f(x,y) = e^x \cos y$	$f(0, \pi/2) = e^0 \cos \pi/2 = 0$
$f_x(x,y) = e^x \cos y$	$f_x(0, \pi/2) = e^0 \cos \pi/2 = 0$
$f_y(x,y) = -e^x \sin y$	$f_y(0, \pi/2) = -e^0 \sin \pi/2 = -1$
$f_{xx}(x,y) = e^x \cos y$	$f_{xx}(0, \pi/2) = e^0 \cos \pi/2 = 0$
$f_{xy}(x,y) = -e^x \sin y$	$f_{xy}(0, \pi/2) = -e^0 \sin \pi/2 = -1$
$f_{yy}(x,y) = -e^x \cos y$	$f_{yy}(0, \pi/2) = -e^0 \cos \pi/2 = 0$
$f_{xxx}(x,y) = e^x \cos y$	$f_{xxx}(0, \pi/2) = e^0 \cos \pi/2 = 0$
$f_{xxy}(x,y) = -e^x \sin y$	$f_{xxy}(0, \pi/2) = -e^0 \sin \pi/2 = -1$
$f_{xyy}(x,y) = -e^x \cos y$	$f_{xyy}(0, \pi/2) = -e^0 \cos \pi/2 = 0$
$f_{yyy}(x,y) = e^x \sin y$	$f_{yyy}(0, \pi/2) = e^0 \sin \pi/2 = 1$

Taylor's series

$$f(x,y) = f(a,b) + [h f_x(a,b) + k f_y(a,b)]$$
$$+ \tfrac{1}{2}[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)]$$
$$+ \tfrac{1}{6}[h^3 f_{xxx}(a,b) + 3h^2k f_{xxy}(a,b) + 3hk^2 f_{xyy}(a,b) + k^3 f_{yyy}(a,b)]$$

$$= 0 + [x(0) + (y-\pi/2)(-1)] + \tfrac{1}{2}[x^2(0) + 2(x)(y-\pi/2)(-1) + (y-\pi/2)^2(0)]$$
$$+ \tfrac{1}{6}[x^3(0) + 3x^2(y-\pi/2)(-1) + 3x(y-\pi/2)^2(0) + (y-\pi/2)^2(1)]$$

$$= 0 + [0 - (y - \pi/2)] + 1/2 [0 + 2x(y - \pi/2) + 0]$$
$$+ 1/6 [0 + 3x^2 (y - \pi/2) + 0 + (y - \pi/2)^2]$$

$$= -y + \pi/2 + 1/2 (-2x)(y - \pi/2) + 1/6 [-3x^2 (y - \pi/2) + (y - \pi/2)^2]$$

$$f(x,y) = -y + \pi/2 - x(y - \pi/2) - 3/6\, x^2 (y - \pi/2) + 1/6 (y - \pi/2)^2$$

Example - (3) Obtain the Taylor's series of $x^3 + y^3 + xy^2$ in powers of $(x-1)$ and $(y-2)$.

Soln: Given that $f(x,y) = x^3 + y^3 + xy^2$ and $\boxed{a=1}$ & $\boxed{b=2}$

Here $\boxed{h = x-1}$ & $\boxed{k = y-2}$

$f(x,y) = x^3 + y^3 + xy^2$	$f(1,2) = (1)^3 + (2)^3 + (1)(2)^2 = 13$
$f_x(x,y) = 3x^2 + y^2$ $f_y(x,y) = 3y^2 + 2xy$	$f_x(1,2) = 3(1)^2 + (2)^2 = 3 + 4 = 7$ $f_y(1,2) = 3(2)^2 + 2(1)(2) = 12 + 4 = 16$
$f_{xx}(x,y) = 6x$ $f_{xy}(x,y) = 2y$ $f_{yy}(x,y) = 6y + 2x$	$f_{xx}(1,2) = 6(1) = 6$ $f_{xy}(1,2) = 2(2) = 4$ $f_{yy}(1,2) = 6(2) + 2(1) = 12 + 2 = 14$
$f_{xxx}(x,y) = 6$ $f_{xxy}(x,y) = 0$ $f_{xyy}(x,y) = 2$ $f_{yyy}(x,y) = 6$	$f_{xxx}(1,2) = 6$ $f_{xxy}(1,2) = 0$ $f_{xyy}(1,2) = 2$ $f_{yyy}(1,2) = 6$

Taylor's series,

$$f(x,y) = f(a,b) + [h f_x(a,b) + k f_y(a,b)] + \frac{1}{2}[h^2 f_{xx}(a,b) + hk f_{xy}(a,b) + k^2 f_{yy}(a,b)]$$
$$+ \frac{1}{6}[h^3 f_{xxx}(a,b) + 3h^2k f_{xxy}(a,b) + 3hk^2 f_{xyy}(a,b) + k^3 f_{yyy}(a,b)]$$

$$= 13 + [(x-1)(7) + (y-2)(16)] + \frac{1}{2}[(x-1)^2(6) + (x-1)(y-2)(4) + (y-2)^2 14]$$
$$+ \frac{1}{6}[(x-1)^3(6) + 3(x-1)^2(y-2)(0) + 3(x-1)(y-2)^2(2) + (y-2)^2(6)].$$

$$= 13 + 7(x-1) + 16(y-2) + \frac{1}{2}[6(x-1)^2 + 4(x-1)(y-2) + 14(y-2)^2]$$
$$+ \frac{1}{6}[6(x-1)^3 + 0 + 6(x-1)(y-2)^2 + 6(y-2)^2]$$

$$f(x,y) = 13 + 7(x-1) + 16(y-2) + 3(x-1)^2 + 2(x-1)(y-2) + 7(y-2)^2$$
$$+ (x-1)^3 + (x-1)(y-2)^2 + (y-2)^2 \;//$$

Example - (4)

Expand $x^2y^2 + 2x^2y + 3xy^2$ in powers of $(x+2)$ and $(y-1)$ using Taylor's series upto third degree terms.

Soln: Given that $f(x,y) = x^2y^2 + 2x^2y + 3xy^2$

and $\boxed{h = x+2}$ & $\boxed{k = y-1}$

Here $\boxed{a = -2}$; $\boxed{b = 1}$

(A1)

	$x=-2,\ y=1$
$f(x,y)=x^2y^2+2x^2y+3xy^2$	$f(-2,1)=(-2)^2(1)^2+2(-2)^2(1)+3(-2)(1)^2$ $=4+8-6=6$
$f_x(x,y)=2xy^2+4xy+3y^2$	$f_x(-2,1)=2(-2)(1)^2+4(-2)(1)+3(1)^2$ $=-4-8+3=-9$
$f_y(x,y)=2x^2y+2x^2+6yx$	$f_y(-2,1)=2(-2)^2(1)+2(-2)^2+6(1)(-2)$ $=8+8-12=4$
$f_{xx}(x,y)=2y^2+4y$	$f_{xx}(-2,1)=2(1)^2+4(1)=2+4=6$
$f_{xy}(x,y)=4yx+4x+6y$	$f_{xy}(-2,1)=4(1)(-2)+4(-2)+6(1)$ $=-8-8+6=-10$
$f_{yy}(x,y)=2x^2+6x$	$f_{yy}(-2,1)=2(-2)^2+6(1)=8+6=14$
$f_{xxx}(x,y)=0$	$f_{xxx}(-2,1)=0$
$f_{xxy}(x,y)=4y+4$	$f_{xxy}(-2,1)=4(1)+4=8$
$f_{xyy}(x,y)=4x+6$	$f_{xyy}(-2,1)=4(-2)+6=-8+6=-2$
$f_{yyy}(x,y)=0$	$f_{yyy}(-2,1)=0$

Taylor's series:

$$f(x,y)=f(a,b)+\left[hf_x(a,b)+kf_y(a,b)\right]+\tfrac{1}{2}\left[h^2f_{xx}(a,b)+2hkf_{xy}(a,b)+k^2f_{yy}(a,b)\right]$$
$$+\tfrac{1}{6}\left[h^3f_{xxx}(a,b)+3h^2kf_{xxy}(a,b)+3hk^2f_{xyy}(a,b)+k^3f_{yyy}(a,b)\right]$$

$$=6+\left[(x+2)(-9)+(y-1)(4)\right]+\tfrac{1}{2}\left[(x+2)^2(6)+2(x+2)(y-1)(-10)+(y-1)^2(14)\right]$$
$$+\tfrac{1}{6}\left[(x+2)^3(0)+3(x+2)^2(y-1)(8)+3(x+2)(y-1)^2(-2)+(y-1)^3(0)\right]$$

$$= 6 + [(-9)(x+2) + 4(y-1)] + \tfrac{1}{2}[6(x+2)^2 - 20(x+2)(y-1) + 14(y-1)^2]$$

$$+ \tfrac{1}{6}[0 + 24(x+2)^2(y-1) - 6(x+2)(y-1)^2 + 0]$$

$$= 6 - 9x - 18 + 4y - 4 + [3(x+2)^2 - 10(x+2)(y-1) + 7(y-1)^2]$$

$$+ 4(x+2)^2(y-1) - (x+2)(y-1)^2$$

$$f(x,y) = -16 - 9x + 4y + 3(x+2)^2 - 10(x+2)(y-1) + 7(y-1)^2$$

$$+ 4(x+2)^2(y-1) - (x+2)(y-1)^2.$$

Example - 5

Expand $e^x \log(1+y)$ in Powers of x and y upto the terms of third degree by using Taylor's series Expansion.

Soln: Given that $f(x,y) = e^x \log(1+y)$ and

$\boxed{h = x - 0}$ & $\boxed{k = y - 0}$

Here $\boxed{a = 0}$ & $\boxed{b = 0}$

is $h = x$ & $k = y$.

$\boxed{e^0 = 1}$ & $\boxed{\log(1) = 0}$

$f(x,y) = e^x \log(1+y)$	$f(0,0) = e^0 \log(1+0) = (1) \log(1) = 1(0) = 0$
$f_x(x,y) = e^x \log(1+y)$ $f_y(x,y) = e^x\left(\frac{1}{1+y}\right)$	$f_x(0,0) = e^0 \log(1+0) = 0$ $f_y(0,0) = e^0\left(\frac{1}{1+0}\right) = (1)\left(\frac{1}{1}\right) = 1$
$f_{xx}(x,y) = e^x \log(1+y)$ $f_{xy}(x,y) = e^x\left(\frac{1}{1+y}\right)$ $f_{yy}(x,y) = -e^x(1+y)^{-2}$	$f_{xx}(0,0) = e^0 \log(1+0) = 0$ $f_{xy}(0,0) = e^0\left(\frac{1}{1+0}\right) = (1)(1) = 1$ $f_{yy}(0,0) = -e^0(1+0)^{-2} = (-1)(1) = -1$
$f_{xxx}(x,y) = e^x \log(1+y)$ $f_{xxy}(x,y) = e^x\left(\frac{1}{1+y}\right)$ $f_{xyy}(x,y) = -e^x(1+y)^{-2}$ $f_{yyy}(x,y) = 2e^x(1+y)^{-3}$	$f_{xxx}(0,0) = e^0 \log(1+0) = 0$ $f_{xxy}(0,0) = e^0\left(\frac{1}{1+0}\right) = (1)(1) = 1$ $f_{xyy}(0,0) = -e^0(1+0)^{-2} = (-1)(1) = -1$ $f_{yyy}(0,0) = 2e^0(1+0)^{-3} = 2(1) = 2.$

Taylor's series

$$f(x,y) = f(a,b) + [h f_x(a,b) + k f_y(a,b)]$$
$$+ \tfrac{1}{2}[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)]$$
$$+ \tfrac{1}{6}[h^3 f_{xxx}(a,b) + 3h^2 k f_{xxy}(a,b) + 3hk^2 f_{xyy}(a,b) + k^3 f_{yyy}(a,b)]$$

$$f(x,y) = 0 + [x(0) + y(1)] + \tfrac{1}{2}[x^2(0) + 2xy(1) + y^2(-1)]$$
$$+ \tfrac{1}{6}[x^3(0) + 3x^2y(1) + 3xy^2(-1) + y^3(2)]$$

$$= 0 + [0 + y] + \tfrac{1}{2}[0 + 2xy - y^2]$$
$$+ \tfrac{1}{6}[0 + 3x^2y - 3xy^2 + 2y^3]$$

$$= y + \tfrac{1}{2}[2xy - y^2] + \tfrac{1}{6}[3x^2y - 3xy^2 + 2y^3]$$

$$= y + xy - \frac{y^2}{2} + \tfrac{1}{2}x^2y - \tfrac{1}{2}xy^2 + \tfrac{1}{3}y^3.$$

$$f(x,y) = y + xy - \frac{y^2}{2} + \frac{x^2y}{2} - \frac{xy^2}{2} + \frac{y^3}{3} \quad //$$

Chapter - 3.6

Maxima And Minima For The Functions of Two Variables

Procedure to find the maxima and minima of $f(x,y)$

Step - 1

Find the f_x & f_y and Equal to zero

i.e, $f_x = 0$ & $f_y = 0$, we get the solution

Point (a, b).

Step - 2

Next to find $A = f_{xx}$; $B = f_{xy}$

and $C = f_{yy}$

Step - 3 To find the value $\nabla = AC - B^2$.

(i) If $\nabla > 0$ and $A < 0$, then $f(x,y)$ is maximum at (a,b)

(ii) If $\nabla > 0$ and $A > 0$, then $f(x,y)$ is minimum at (a,b)

(iii) If $\nabla < 0$, then $f(x,y)$ is saddle Point

(iv) If $\nabla = 0$, then nothing is known and further investigation is required.

Example - ①

Find the maxima and minima of

$f(x,y) = x^3 + y^3 - 3x - 12y + 20.$

Soln.

Given that $f(x,y) = x^3 + y^3 - 3x - 12y + 20.$

To find stationary Points :-

$f_x = 3x^2 - 3$ & $f_y = 3y^2 - 12$

$f_x = 0 \Rightarrow 3x^2 - 3 = 0$

$3x^2 = 3$

$x^2 = 3/3$

$x^2 = 1$

$\boxed{x = \pm 1}$

& $\boxed{x = 1, -1}$

$f_y = 0 \Rightarrow 3y^2 - 12 = 0$

$3y^2 = 12$

$y^2 = 12/3$

$y^2 = 4$

$\boxed{y = \pm 2}$

$\boxed{y = 2, -2}$

The stationary Points are

$(1,2), (1,-2), (-1,2), (-1,-2)$

and

$f_{xx} = 6x$ | $f_{xy} = 0$ | $f_{yy} = 6y$

$\boxed{A = 6x}$ | $\boxed{B = 0}$ | $\boxed{C = 6y}$

Points	$A = 6x$	$B = 0$	$C = 6y$	$\nabla = AC - B^2$
$(1, 2)$	$A = 6(1)$ $A = 6$	$B = 0$	$C = 6(2)$ $C = 12$	$\nabla = 6(12) - 0$ $\nabla = 72 > 0$
$(1, -2)$	$A = 6(1)$ $A = 6$	$B = 0$	$C = 6(-2)$ $= -12$	$\nabla = 6(-12) - 0$ $\nabla = -72 < 0$
$(-1, 2)$	$A = 6(-1)$ $= -6$	$B = 0$	$C = 6(2)$ $= 12$	$\nabla = (-6)(12) - 0$ $\nabla = -72 < 0$
$(-1, -2)$	$A = 6(-1)$ $= -6$	$B = 0$	$C = 6(-2)$ $= -12$	$\nabla = (-6)(-12) - 0$ $\nabla = 72 > 0$

Here, the Point $(1,-2)$ $(-1,2)$ have $\nabla < 0$.

$\therefore$ It is Saddle Points

(1) The Point $(1,2)$ have $\nabla > 0$ & $A > 0$.

$\therefore$ It is minimum Point

$$f(x,y) = x^3 + y^3 - 3x - 12y + 20$$

$$f(1,2) = (1)^3 + (2)^3 - 3(1) - 12(2) + 20$$

$$= 1 + 8 - 3 - 24 + 20$$

$\boxed{f(1,2) = 2}$ is minimum value.

(ii) The Point $(-1,-2)$ have $\nabla > 0$ & $A < 0$.

$\therefore$ It is maximum Point.

$$f(x,y) = x^3 + y^3 - 3x - 12y + 20$$

$$f(-1,-2) = (-1)^3 + (-2)^3 - 3(-1) - 12(-2) + 20$$

$$= -1 - 8 + 3 + 24 + 20$$

$\boxed{f(-1,-2) = 38}$ is maximum value.

Example-②

Find the minimum Point of $f(x,y) = x^2 + y^2 + 6x + 12$.

Soln:

Given that $f(x,y) = x^2 + y^2 + 6x + 12$.

To find stationary Points:-

$f_x = 2x + 6$ & $f_y = 2y$

$f_x = 0 \Rightarrow 2x + 6 = 0$

$2x = -6$

$x = -6/2$

$\boxed{x = -3}$

$f_y = 0 \Rightarrow 2y = 0$

$\boxed{y = 0}$

The Points is $(-3, 0)$

then, $f_{xx} = 2$, $\boxed{A = 2}$ | $f_{xy} = 0$, $\boxed{B = 0}$ | $f_{yy} = 2$, $\boxed{C = 2}$

At the Point $(-3,0)$

$$\nabla = AC - B^2$$
$$= 2(2) - 0$$
$$\boxed{\nabla = 4} > 0 \quad \text{and} \quad \boxed{A = 2} > 0$$

Hence the point $(-3,0)$ is minimum point.

$$f(x,y) = x^2 + y^2 + 6x + 12$$
$$f(-3,0) = (-3)^2 + 0 + 6(-3) + 12$$
$$= 9 - 18 + 12$$
$$= 21 - 18$$
$$\boxed{f(-3,0) = 3} \text{ is minimum value.}$$

Example - 3

Examine $f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$ for extreme values.

Soln: Given that $f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$.

To find stationary Points:

$$f_x = 3x^2 + 3y^2 - 30x + 72 \quad \& \quad f_y = 6yx - 30y$$
$$f_x = 0 \Rightarrow 3x^2 + 3y^2 - 30x + 72 = 0$$
$$3(x^2 + y^2 - 10x + 24) = 0. \longrightarrow (1)$$

and $f_y = 0 \Rightarrow 6xy - 30y = 0$

$$6y(x-5) = 0 \longrightarrow (2)$$

from (2) $\Rightarrow 6y(x-5) = 0$

$6y = 0$ and $x - 5 = 0$

$\boxed{y=0}$ and $\boxed{x=5}$

when $\boxed{y=0}$ in Eqn (1)

(1) $\Rightarrow x^2 + y^2 - 10x + 24 = 0$

$x^2 + 0 - 10x + 24 = 0$

$x^2 - 10x + 24 = 0$

$(x-4)(x-6) = 0$

$x - 4 = 0$ | $x - 6 = 0$

$\boxed{x=4}$ | $\boxed{x=6}$

when $\boxed{x=5}$ in Eqn (1)

(1) $\Rightarrow x^2 + y^2 - 10x + 24 = 0$

$(5)^2 + y^2 - 10(5) + 24 = 0$

$25 + y^2 - 50 + 24 = 0$

$y^2 - 1 = 0$

$y^2 = 1$

$\boxed{y = \pm 1}$

$\therefore$ The stationary Points are $(4,0), (6,0), (5,1) (5,-1)$

then $f_{xx} = 6x-30$; $f_{xy} = 6y$; $f_{yy} = 6x-30$

$A = 6x-30$; $B = 6y$; $C = 6x-30$

Points	$A = 6x-30$	$B = 6y$	$C = 6x-30$	$\nabla = AC - B^2$
$(4,0)$	$A = 6(4)-30$ $= 24-30$ $\boxed{A=-6}$	$\boxed{B=0}$	$C = 6(4)-30$ $= 24-30$ $\boxed{C=-6}$	$\nabla = (-6)(-6) - 0$ $= 36 - 0$ $\boxed{\nabla = 36} > 0$
$(6,0)$	$A = 6(6)-30$ $= 36-30$ $\boxed{A=6}$	$\boxed{B=0}$	$C = 6(6)-30$ $= 36-30$ $\boxed{C=6}$	$\nabla = (6)(6) - 0$ $= 36-0$ $\boxed{\nabla = 36} > 0$
$(5,1)$	$A = 6(5)-30$ $= 30-30$ $\boxed{A=0}$	$B = 6(1)$ $\boxed{B=6}$	$C = 6(5)-30$ $= 30-30$ $\boxed{C=0}$	$\nabla = 0-(6)^2$ $\boxed{\nabla = -36} < 0$
$(5,-1)$	$A = 6(5)-30$ $= 30-30$ $\boxed{A=0}$	$B = 6(-1)$ $\boxed{B=-6}$	$C = 6(5)-30$ $= 30-30$ $\boxed{C=0}$	$\nabla = 0-(-6)^2$ $\boxed{\nabla = -36} < 0$

Here the Points $(5,-1)$, $(5,1)$ have $\nabla < 0$

$\therefore$ It is saddle Points

(i) The Point $(4,0)$ have $\nabla > 0$ and $A < 0$

$\therefore$ It is maximum Point

$$f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x.$$

$$f(4,0) = (4)^3 + 3(4)(0) - 15(4)^2 - 15(0) + 72(4)$$

$$= 64 + 0 - 15(16) - 0 + 288$$

$$= 64 - 240 + 288$$

$\boxed{f(4,0) = 112}$ is maximum value.

(ii) The Point $(6,0)$ have $\nabla > 0$ and $A > 0$

$\therefore$ It is minimum Point

$$f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$$

$$f(6,0) = (6)^3 + 3(6)(0) - 15(6)^2 - 15(0) + 72(6)$$

$$= 216 + 0 - 15(36) - 0 + 432$$

$$= 216 - 540 + 432$$

$\boxed{f(6,0) = 108}$ is minimum value.

Example - 4

Find the maximum or minimum values of $f(x,y) = 3x^2 - y^2 + x^3$.

Soln:

Given that $f(x,y) = 3x^2 - y^2 + x^3$

To find stationary points :-

$f_x = 6x + 3x^2$ and $f_y = -2y$

$f_x = 0 \Rightarrow 6x + 3x^2 = 0$

$3x(2+x) = 0$

$3x = 0 ;\ 2+x = 0$

$\boxed{x=0}$; $\boxed{x=-2}$

$f_y = 0 \Rightarrow -2y = 0$

$\boxed{y=0}$

$\therefore$ The points are $(0,0), (-2,0)$

and $f_{xx} = 6+6x$ | $f_{xy} = 0$ | $f_{yy} = -2$

$\boxed{A = 6x+6}$ | $\boxed{B=0}$ | $\boxed{C=-2}$

Points	$A = 6x+6$	$B=0$	$C=-2$	$\nabla = AC - B^2$
(i) $(0,0)$	$A = 0+6$ $\boxed{A=6}$	$\boxed{B=0}$	$\boxed{C=-2}$	$\nabla = 6(-2) - 0$ $\nabla = -12 < 0$
(ii) $(-2,0)$	$A = 6(-2)+6$ $= -12+6$ $\boxed{A=-6}$	$\boxed{B=0}$	$\boxed{C=-2}$	$\nabla = -6(-2) - 0$ $\nabla = 12 > 0$

(i) The point $(0,0)$ have the value $\nabla < 0$

$\therefore$ It is saddle point

ii) The point $(-2,0)$ have $\nabla > 0$ and $A < 0$

$\therefore$ It is maximum point

$$f(x,y) = 3x^2 - y^2 + x^3$$

$$f(-2,0) = 3(-2)^2 - (0) + (-2)^3$$

$$= 3(4) - 8$$

$$= 12 - 8$$

$$\boxed{f(-2,0) = 4}$$

Example-5 Find the maxima and minima value for $f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$.

Soln:

Given that $f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

To find the stationary point:

$$f_x = 4x^3 - 4x + 4y \quad \& \quad f_y = 4y^3 + 4x - 4y$$

$f_x = 0 \Rightarrow 4x^3 - 4x + 4y = 0$

$4(x^3 - x + y) = 0$

$x^3 - x + y = 0 \rightarrow (1)$

$f_y = 0 \Rightarrow 4y^3 + 4x - 4y = 0$

$4(y^3 + x - y) = 0$

$y^3 + x - y = 0 \rightarrow (2)$

Adding (1) & (2) $\quad (1) + (2) \Rightarrow x^3 - \cancel{x} + \cancel{y} + y^3 + \cancel{x} - \cancel{y} = 0$

$$x^3 + y^3 = 0$$

$$x^3 = -y^3$$

$$\boxed{x = -y} \rightarrow (3)$$

Put $\boxed{x=-y}$ in Eqn ②

$$② \Rightarrow y^3+x-y=0$$
$$y^3-y-y=0$$
$$y^3-2y=0$$
$$y(y^2-2)=0$$
$$\boxed{y=0};\quad y^2-2=0$$
$$y^2=2$$
$$\boxed{y=\pm\sqrt{2}}$$

Put $\boxed{y=-x}$ in Eqn ①

$$① \Rightarrow x^3-x+y=0$$
$$x^3-x-x=0$$
$$x^3-2x=0$$
$$x(x^2-2)=0$$
$$\boxed{x=0};\quad x^2-2=0$$
$$x^2=2$$
$$\boxed{x=\pm\sqrt{2}}$$

The Points are, $(0,0)$ $(\sqrt{2},-\sqrt{2})$, $(-\sqrt{2},\sqrt{2})$

then $f_{xx}=12x^2-4$ | $f_{xy}=4$ | $f_{yy}=12y^2-4$

$\boxed{A=12x^2-4}$ | $\boxed{B=4}$ | $\boxed{C=12y^2-4}$

Points	$A=12x^2-4$	$B=4$	$C=12y^2-4$	$\nabla=AC-B^2$
(i) $(0,0)$	$A=0-4$ $\boxed{A=-4}$	$\boxed{B=4}$	$C=0-4$ $\boxed{C=-4}$	$\nabla=(-4)(-4)-(4)^2$ $\nabla=16-16=0$ $\nabla=0$
(ii) $(\sqrt{2},-\sqrt{2})$	$A=12(\sqrt{2})^2-4$ $=12(2)-4$ $=24-4$ $\boxed{A=20}$	$\boxed{B=4}$	$C=12(-\sqrt{2})^2-4$ $=12(2)-4$ $=24-4$ $\boxed{C=20}$	$\nabla=(20)(20)-(4)^2$ $=400-16$ $\nabla=384>0$
(iii) $(-\sqrt{2},\sqrt{2})$	$A=12(-\sqrt{2})^2-4$ $=12(2)-4$ $=24-4$ $\boxed{A=20}$	$\boxed{B=4}$	$C=12(\sqrt{2})^2-4$ $=12(2)-4$ $=24-4$ $\boxed{C=20}$	$\nabla=(20)(20)-(4)^2$ $=400-16$ $\nabla=384>0$

The Point $(0,0)$ have the value $\nabla = 0$

$\therefore$ It is saddle Point.

(i) The Point $(\sqrt{2}, -\sqrt{2})$ have the value $\nabla > 0$ and $A > 0$

$\therefore$ It is minimum point

$$f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$$

$$f(\sqrt{2},-\sqrt{2}) = (\sqrt{2})^4 + (-\sqrt{2})^4 - 2(\sqrt{2})^2 + 4(\sqrt{2})(-\sqrt{2}) - 2(-\sqrt{2})^2$$

$$= 4 + 4 - 2(2) + 4(-2) - 2(2)$$

$$= \cancel{8} - 4 - \cancel{8} - 4$$

$$\boxed{f(\sqrt{2},-\sqrt{2}) = -8}$$

(ii) The point $(-\sqrt{2}, \sqrt{2})$ have the value of $\nabla > 0$ and $A > 0$

$\therefore$ It is minimum Point

$$f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$$

$$f(-\sqrt{2},\sqrt{2}) = (-\sqrt{2})^4 + (\sqrt{2})^4 - 2(-\sqrt{2})^2 + 4(-\sqrt{2})(\sqrt{2}) - 2(-\sqrt{2})^2$$

$$= 4 + 4 - 2(2) + 4(-2) - 2(2)$$

$$= \cancel{8} - 4 - \cancel{8} - 4$$

$$\boxed{f(-\sqrt{2},\sqrt{2}) = -8}$$

Example-⑥ Discuss the maxima and minima value of $f(x,y) = x^3y^2(1-x-y)$.

Soln Given $f(x,y) = x^3y^2(1-x-y)$

$$f(x,y) = x^3y^2 - x^4y^2 - x^3y^3$$

To find stationary Points:-

$$f_x = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$$

$$f_x = 0 \Rightarrow 3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$$

$$x^2y^2(3-4x-3y) = 0$$

$$3-4x-3y = 0$$

$$4x+3y-3 = 0 \longrightarrow ①$$

$$f_y = 2yx^3 - 2yx^4 - 3y^2x^3$$

$$f_y = 0 \Rightarrow 2yx^3 - 2yx^4 - 3y^2x^3 = 0$$

$$yx^3(2-2x-3y) = 0$$

$$2-2x-3y = 0$$

$$2x+3y-2 = 0 \longrightarrow ②$$

$$① \Rightarrow 4x + \cancel{3y} - 3 = 0$$

$$② \Rightarrow \underset{(-)}{2x} + \underset{(-)}{\cancel{3y}} \underset{(+)}{-2} = 0$$

$$2x - 1 = 0$$

$$2x = 1$$

$$\boxed{x = 1/2}$$

Put $\boxed{x = 1/2}$ in Eqn ②

② $\Rightarrow$ $2x + 3y - 2 = 0$

$\cancel{2}(1/\cancel{2}) + 3y - 2 = 0$

$1 + 3y - 2 = 0$

$3y - 1 = 0$

$3y = 1 \Rightarrow \boxed{y = 1/3}$

The Point is $(1/2, 1/3)$.

$A = f_{xx} = 6xy^2 - 12x^2y^2 - 6xy^3$

$A = \cancel{6}^{3}(1/\cancel{2})(1/3)^2 - 12(1/2)^2(1/3)^2 - 6(1/2)(1/3)^3$

$= \cancel{3}(1/\cancel{9}_{3}) - \cancel{12}^{3}(1/\cancel{4})(1/\cancel{9}_{3}) - \cancel{6}^{3}(1/\cancel{2})(1/\cancel{27}_{9})$

$= \cancel{1/3} - \cancel{1/3} - 1/9$

$\boxed{A = -1/9} < 0$

$B = f_{xy} = 6x^2y - 8x^3y - 9x^2y^2$

$B = 6(1/2)^2(1/3) - 8(1/2)^3(1/3) - 9(1/2)^2(1/3)^2$

$= \cancel{6}^{2}(1/\cancel{4}_{2})(1/\cancel{3}) - \cancel{8}(1/\cancel{8})(1/3) - \cancel{9}(1/4)(1/\cancel{9})$

$= 1/2 - 1/3 - 1/4$

$\boxed{B = -1/12}$

$C = f_{yy} = 2x^3 - 2x^4 - 6x^3y$

$C = 2(1/2)^3 - 2(1/2)^4 - 6(1/2)^3(1/3)$

$= \cancel{2}(1/\cancel{8}_4) - \cancel{2}(1/\cancel{16}_8) - \cancel{6}^2(1/\cancel{8}_4)(1/\cancel{3})$

$= \cancel{1/4} - 1/8 - \cancel{1/4}$

$\boxed{C = -1/8}$

To find:

$\nabla = AC - B^2$

$= (-1/9)(-1/8) - (-1/12)^2$

$= \frac{1}{72} - \frac{1}{144}$

$\boxed{\nabla = \frac{1}{144}} > 0$

The Point $(1/2, 1/3)$ have the value of $\nabla > 0$ & $A < 0$

$\therefore$ It is maximum Point.

$f(x,y) = x^3y^2(1-x-y)$

$f(1/2, 1/3) = (1/2)^3(1/3)^2(1 - 1/2 - 1/3)$

$= (1/8)(1/9)\left[\frac{6-3-2}{6}\right]$

$= \frac{1}{72}(1/6)$

$\boxed{f(1/2, 1/3) = \frac{1}{432}}$

Chapter-3.7

Method of Lagrangian multiplier

We define $F(x,y,z) = f(x,y,z) + \lambda\, g(x,y,z)$

The necessary conditions for maximum or minimum

$$\frac{\partial F}{\partial x} = 0 \; ; \; \frac{\partial F}{\partial y} = 0 ; \; \frac{\partial F}{\partial z} = 0$$

$$F_x = 0 \; ; \; F_y = 0 \; ; \; F_z = 0.$$

Example-①

Find the dimensions of the rectangular box without a top of maximum capacity, whose surface area is 108 sq.cms.

Soln:

Let the given surface area as

$$g(x,y,z) = xy + 2xz + 2yz = 108 \longrightarrow (A)$$

$$g(x,y,z) = xy + 2xz + 2yz - 108$$

The volume is $f(x,y,z) = xyz$

Let us consider the Lagrangian function as

$$F(x,y,z) = f(x,y,z) + \lambda\, g(x,y,z)$$

$$F(x,y,z) = xyz + \lambda(xy + 2xz + 2yz - 108) \longrightarrow (B)$$

$$\frac{\partial F}{\partial x} = yz + \lambda(y+2z) \; ; \; \frac{\partial F}{\partial y} = xz + \lambda(x+2z) \; ; \; \frac{\partial F}{\partial z} = xy + \lambda(2x+2y)$$

$$\frac{\partial F}{\partial x} = 0 \Rightarrow yz + \lambda(y+2z) = 0$$

$$\Rightarrow yz = -\lambda(y+2z)$$

$$\not\Rightarrow \cancel{yz = -\lambda y - 2\lambda z}$$

$$\Rightarrow -\frac{1}{\lambda} = \frac{y+2z}{yz} = \frac{\cancel{y}}{\cancel{y}z} + \frac{2\cancel{z}}{y\cancel{z}}$$

$$\boxed{-\frac{1}{\lambda} = \frac{1}{z} + \frac{2}{y}} \longrightarrow (1)$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow xz + \lambda(x+2z)$$

$$\Rightarrow xz = -\lambda(x+2z)$$

$$\Rightarrow -\frac{1}{\lambda} = \frac{x+2z}{xz} = \frac{\cancel{x}}{\cancel{x}z} + \frac{2\cancel{z}}{x\cancel{z}}$$

$$\Rightarrow \boxed{-\frac{1}{\lambda} = \frac{1}{z} + \frac{2}{x}} \longrightarrow (2)$$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow xy + \lambda(2x+2y)$$

$$\Rightarrow xy = -\lambda(2x+2y)$$

$$\Rightarrow -\frac{1}{\lambda} = \frac{2x+2y}{xy} = \frac{2\cancel{x}}{\cancel{x}y} + \frac{2\cancel{y}}{x\cancel{y}}$$

$$\Rightarrow \boxed{-\frac{1}{\lambda} = \frac{2}{y} + \frac{2}{x}} \longrightarrow (3)$$

from (1) & (2)

$$\cancel{\frac{1}{z}} + \frac{2}{y} = \cancel{\frac{1}{z}} + \frac{2}{x}$$

$$\frac{2}{y} = \frac{2}{x}$$

$$2x = 2y$$

$$\boxed{x = y} \longrightarrow (4)$$

from (2) & (3)

$$\frac{1}{z} + \cancel{\frac{2}{x}} = \frac{2}{y} + \cancel{\frac{2}{x}}$$

$$\frac{1}{z} = \frac{2}{y}$$

$$\boxed{y = 2z} \longrightarrow (5)$$

from (4) & (5)

$$x = y = 2z \longrightarrow (6)$$

Eqn (A) $\Rightarrow$ $xy + 2xz + 2yz = 108$

$$(2z)(2z) + 2(2z)z + 2(2z)z = 108$$

$$4z^2 + 4z^2 + 4z^2 = 108$$

$$12z^2 = 108$$

$$z^2 = \frac{108}{12} \Rightarrow z^2 = 9$$

$$\boxed{z = \pm 3}$$

$\therefore$ $\boxed{z=3}$

ie, $x = 2z \Rightarrow x = 2(3) \Rightarrow \boxed{x=6}$

$y = 2z \Rightarrow y = 2(3) \Rightarrow \boxed{y=6}$

The dimensions of the box, $x=6$, $y=6$, $z=3$

<u>Example-(2)</u> A thin closed rectangular box is to have one edge equal to twice the other and constant volume is $72\ m^3$. Find the least surface area of the box.

<u>Soln:</u>

Let $x, y, 2y$ be the length, breadth and height of the box respectively.

$$\text{surface area} = 2(x)y + 2y(2y) + 2x(2y)$$

$$= 2xy + 4y^2 + 4xy$$

$$f(x,y,z) = 6xy + 4y^2 \longrightarrow (A)$$

Volume (V) $\Rightarrow xyz = 72$

$$xy(2y) = 72$$

$$2xy^2 = 72$$

$$xy^2 = 72/2$$

$$g(x,y,z) = xy^2 = 36$$

$$g(x,y,z) = xy^2 - 36 \longrightarrow \text{(B)}$$

Let us consider the Lagrangian function is

$$F(x,y,z) = f(x,y,z) + \lambda g(x,y,z)$$

$$F(x,y,z) = (6xy + 4y^2) + \lambda(xy^2 - 36)$$

$$\frac{\partial F}{\partial x} = 6y + \lambda y^2$$

$$\frac{\partial F}{\partial x} = 0 \Rightarrow 6y + \lambda y^2 = 0$$

$$6\cancel{y} = -\lambda y^{\cancel{2}}$$

$$6 = -\lambda y$$

$$\boxed{\frac{6}{y} = -\lambda} \longrightarrow (1)$$

$$\frac{\partial F}{\partial y} = 6x + 8y + 2\lambda xy$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow 6x + 8y + 2\lambda xy = 0$$

$$6x + 8y = -2\lambda xy$$

$$3x + 4y = -\lambda xy$$

$$\frac{3x + 4y}{xy} = -\lambda$$

$$\frac{3\cancel{x}}{\cancel{x}y} + \frac{4\cancel{y}}{x\cancel{y}} = -\lambda$$

$$\boxed{\frac{3}{y} + \frac{4}{x} = -\lambda} \longrightarrow (2)$$

from ① & ②, we get

$$\frac{6}{y} = \frac{3}{y} + \frac{4}{x}$$

$$\frac{6}{y} - \frac{3}{y} = \frac{4}{x}$$

$$\frac{3}{y} = \frac{4}{x}$$

$$3x = 4y \Rightarrow 4y = 3x$$

$$\boxed{y = \tfrac{3}{4}x} \longrightarrow ③$$

$$⑬ \Rightarrow xy^2 = 36$$

$$x\left(\tfrac{3}{4}x\right)^2 = 36$$

$$(x)\frac{9}{16}x^2 = 36$$

$$x^3 = \frac{\cancel{36}^{4} \times 16}{\cancel{9}}$$

$$x^3 = 4 \times 16 \Rightarrow x^3 = 64$$

$$x^3 = (4)^3$$

$$\boxed{x = 4}$$

$\cancel{③ \Rightarrow y = \tfrac{3}{4}(4)}$ $③ \Rightarrow y = \tfrac{3}{4}(x)$

$$y = \tfrac{3}{\cancel{4}}(\cancel{4}) \Rightarrow \boxed{y = 3}$$

f is minimum at (4,3)

The minimum surface $= 6xy + 4y^2$

$$= 6(4)(3) + 4(3)^2$$

$$= 72 + 4(9)$$

$$= 72 + 36$$

$$= 108$$

Example - 3

Find the dimensions of the rectangular box without top of maximum capacity with surface area 432 sq.m.

Soln:

Let x, y, z be the length, breadth and height of the box.

surface are $= xy + 2yz + 2zx = 432 \longrightarrow (A)$

$$g(x,y,z) = xy + 2yz - 2zx - 432$$

volume, $f(x,y,z) = xyz \longrightarrow (B)$

Let us consider the Lagrangian function is

$$F(x,y,z) = f(x,y,z) + \lambda g(x,y,z)$$

$$F(x,y,z) = xyz + \lambda(xy + 2yz + 2zx - 432)$$

$$\frac{\partial F}{\partial x} = yz + \lambda(y+2z)$$

$$\frac{\partial F}{\partial x} = 0$$

$$\Rightarrow yz + \lambda(y+2z) = 0$$

$$yz = -\lambda(y+2z)$$

$$-1/\lambda = \frac{y+2z}{yz}$$

$$= \frac{\cancel{y}}{\cancel{y}z} + \frac{2\cancel{z}}{y\cancel{z}}$$

$$\boxed{-1/\lambda = \frac{1}{z} + \frac{2}{y}} \rightarrow (1)$$

$$\frac{\partial F}{\partial y} = zx + \lambda(x+2z)$$

$$\frac{\partial F}{\partial y} = 0$$

$$\Rightarrow zx + \lambda(x+2z) = 0$$

$$zx = -\lambda(x+2z)$$

$$-1/\lambda = \frac{x+2z}{zx}$$

$$= \frac{\cancel{x}}{z\cancel{x}} + \frac{2\cancel{z}}{\cancel{z}x}$$

$$-1/\lambda = \frac{1}{z} + \frac{2}{x}$$

$$\boxed{-1/\lambda = \frac{1}{z} + 2/x} \rightarrow (2)$$

$$\frac{\partial F}{\partial z} = xy + \lambda(2y+2x)$$

$$\frac{\partial F}{\partial z} = 0$$

$$\Rightarrow xy + \lambda(2y+2x) = 0$$

$$xy = -\lambda(2y+2x)$$

$$-1/\lambda = \frac{2y+2x}{xy}$$

$$= \frac{2\cancel{y}}{x\cancel{y}} + \frac{2\cancel{x}}{\cancel{x}y}$$

$$\boxed{-1/\lambda = \frac{2}{x} + \frac{2}{y}} \rightarrow (3)$$

from 1 & 2

$$\cancel{\frac{1}{z}}+\frac{2}{y}=\cancel{\frac{1}{z}}+\frac{2}{x}$$

$$\frac{2}{y}=\frac{2}{x}$$

$$\cancel{2}x=\cancel{2}y$$

$$\boxed{x=y} \longrightarrow (4)$$

from 2 & 3

$$\frac{1}{z}+\cancel{\frac{2}{x}}=\frac{2}{y}+\cancel{\frac{2}{x}}$$

$$\frac{1}{z}=\frac{2}{y}$$

$$\boxed{y=2z} \longrightarrow (5)$$

from 4 & 5 $\quad x=y=2z \longrightarrow (6)$

Eqn (A) $\Rightarrow \quad xy+2yz+2zx=432$

$$(2z)(2z)+2(2z)z+2z(2z)=432$$

$$4z^2+4z^2+4z^2=432$$

$$12z^2=432$$

$$z^2=\frac{432}{12}=36$$

$$z^2=36$$

$$\boxed{z=\pm 6}$$

$\therefore \boxed{z=6}$

by Eqn (6) $\quad x=2z \; ; \; y=2z \; ; \; z=6$

$$x=2(6) \; ; \; y=2(6) \; ; \; \boxed{z=6}$$

$$\boxed{x=12} \; ; \; \boxed{y=12}$$

The dimension of the box $x=12, y=12, z=6$

maximum volume $=xyz=(12)(12)(6)$

$= 864$ cubic meters.

Example (4)

Find the shortest and the longest distance from the point $(1,2,-1)$ to the sphere $x^2+y^2+z^2=24$, using Lagrange's method.

Soln:

Let (x,y,z) be any point on the sphere.

Distance of the point (x,y,z) from $(1,2,-1)$ is

$$d^2 = (x-1)^2+(y-2)^2+(z+1)^2$$

$$f(x,y,z) = (x-1)^2+(y-2)^2+(z+1)^2 \longrightarrow (A)$$

subject to constraint.

$$g(x,y,z) = x^2+y^2+z^2=24 \longrightarrow (B)$$

$$g(x,y,z) = x^2+y^2+z^2-24=0$$

Let us consider the Lagrangian function is

$$F(x,y,z) = f(x,y,z)+\lambda g(x,y,z)$$

$$F(x,y,z) = (x-1)^2+(y-2)^2+(z+1)^2+\lambda(x^2+y^2+z^2-24)$$

$$\frac{\partial F}{\partial x} = 2(x-1)+2\lambda x$$

$$\frac{\partial F}{\partial x}=0$$

$$\cancel{2}(x-1)+\cancel{2}\lambda x=0$$

$$x-1+\lambda x=0$$

$$(1+\lambda)x-1=0$$

$$(1+\lambda)x=1$$

$$\boxed{x=\frac{1}{1+\lambda}} \to (1)$$

$$\frac{\partial F}{\partial y} = 2(y-2)+2\lambda y$$

$$\frac{\partial F}{\partial y}=0$$

$$\cancel{2}(y-2)+\cancel{2}\lambda y=0$$

$$y-2+\lambda y=0$$

$$y+\lambda y-2=0$$

$$(1+\lambda)y-2=0$$

$$(1+\lambda)y=2$$

$$\boxed{\frac{y}{2}=\frac{1}{1+\lambda}} \to (2)$$

$$\frac{\partial F}{\partial z} = 2(z+1)+2\lambda z$$

$$\frac{\partial F}{\partial z}=0$$

$$\cancel{2}(z+1)+\cancel{2}\lambda z=0$$

$$z+1+\lambda z=0$$

$$z+\lambda z+1=0$$

$$(1+\lambda)z+1=0$$

$$(1+\lambda)z=-1$$

$$\boxed{-z=\frac{+1}{1+\lambda}} \to (3)$$

from ① & ③

$\boxed{x = -z} \rightarrow ④$

from ② & ③

$\frac{y}{2} = -z$

$\boxed{y = -2z} \longrightarrow ⑤$

Equ ⑬ $\Rightarrow x^2+y^2+z^2 = 24$

$(-z)^2+(-2z)^2+z^2 = 24$

$z^2+4z^2+z^2 = 24$

$6z^2 = 24$

$z^2 = 24/6 \Rightarrow z^2 = 4 \Rightarrow \boxed{z = \pm 2}$

$\therefore \boxed{z = 2, -2}$

Equ ④ $\Rightarrow \boxed{x = -z}$

If $z = 2$, then $x = -2$

If $z = -2$ then $x = 2$

Equ ⑤ $\Rightarrow \boxed{y = -2z}$

If $z = 2$, then $y = -4$

If $z = 2$, then $y = 4$.

$\therefore$ The stationary points are, $(2,4,-2), (-2,-4,2)$

If the Point $(2,4,-2)$, then $d = \sqrt{(x-1)^2+(y-2)^2+(z+1)^2}$

$= \sqrt{(2-1)^2+(4-2)^2+(-2+1)^2}$

$= \sqrt{(1)^2+(2)^2+(-1)^2}$

$= \sqrt{1+4+1} = \sqrt{6}$ //

If the Point $(-2,-4,2)$, then $d = \sqrt{(-2-1)^2+(-4-2)^2+(2+1)^2}$

$= \sqrt{(-3)^2+(-6)^2+(3)^2}$

$= \sqrt{9+36+9}$

$= \sqrt{54}$ //

Example - 5

Find the minimum values of x^2yz^3 subject to the condition $2x+y+3z=a$.

Soln:

Let $f(x,y,z) = x^2yz^3$ and

$$g(x,y,z) = 2x+y+3z-a \longrightarrow (A)$$

Let us consider the Lagrangian function is

$$F(x,y,z) = f(x,y,z) + \lambda\, g(x,y,z)$$

$$F(x,y,z) = x^2yz^3 + \lambda(2x+y+3z-a)$$

$$\frac{\partial F}{\partial x} = 2xyz^3 + 2\lambda$$

$$\frac{\partial F}{\partial x} = 0$$

$$\cancel{2}xyz^3 + \cancel{2}\lambda = 0$$

$$xyz^3 + \lambda = 0$$

$$\boxed{xyz^3 = -\lambda} \rightarrow (1)$$

$$\frac{\partial F}{\partial y} = x^2z^3 + \lambda$$

$$\frac{\partial F}{\partial y} = 0$$

$$x^2z^3 + \lambda = 0$$

$$\boxed{x^2z^3 = -\lambda} \rightarrow (2)$$

$$\frac{\partial F}{\partial z} = 3z^2x^2y + 3\lambda$$

$$\frac{\partial F}{\partial z} = 0$$

$$3z^2x^2y + 3\lambda = 0$$

$$\cancel{3}z^2x^2y = -\cancel{3}\lambda$$

$$\boxed{z^2x^2y = -\lambda} \rightarrow (3)$$

from (1) & (2)

$$xy\cancel{z^3} = x^2\cancel{z^3}$$

$$\cancel{x}y = x^{\cancel{2}}$$

$$\boxed{y = x} \rightarrow (4)$$

from (2) & (3)

$$\cancel{x^2}z^3 = z^2\cancel{x^2}y$$

$$z^{\cancel{3}} = \cancel{z^2}y$$

$$\boxed{z = y} \rightarrow (5)$$

from (4) & (5)

$$\boxed{x = y = z} \longrightarrow (6)$$

use Eqn (6) in (A)

(A) $\Rightarrow$ $2x+y+3z=a$

$$2x+x+3x=a$$

$$6x=a$$

$$\boxed{x=a/6}$$

(6) $\Rightarrow$ $\boxed{y=a/6}$ and $\boxed{z=a/6}$

∴ the stationary points are $(a/6, a/6, a/6)$

Hence, minimum value $f=x^2yz^3$

$$=(a/6)^2(a/6)(a/6)^3$$

$$=(a/6)^6 \text{ //}$$

Example-(6)

Find the maximum volume of the lagrest rectangular parallelepiped that can be inscribed in an Ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$

Soln

Let volume $f(x,y,z)=8xyz \longrightarrow$ (A)

and $g(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \longrightarrow$ (B)

ie, $g(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1 \longrightarrow$ (B)

Let us consider the Lagrangian function as

$$F(x,y,z) = f(x,y,z) + \lambda g(x,y,z)$$

$$F(x,y,z) = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$$

$$\frac{\partial F}{\partial x} = 8yz + \frac{2x\lambda}{a^2}$$

$$\frac{\partial F}{\partial x} = 0$$

$$\Rightarrow 8yz + \frac{2x\lambda}{a^2} = 0$$

(x) by $x/2$ $\Rightarrow \frac{\cancel{8}^{4}xyz}{2} + \frac{\cancel{2}x^2\lambda}{\cancel{2}a^2}$

$$4xyz + \frac{x^2\lambda}{a^2}$$

$$4xyz = \frac{-x^2\lambda}{a^2}$$

$$\boxed{\frac{4xyz}{-\lambda} = \frac{x^2}{a^2}} \rightarrow (1)$$

$$\frac{\partial F}{\partial y} = 8xz + \frac{2y\lambda}{b^2}$$

$$\frac{\partial F}{\partial y} = 0$$

$$\Rightarrow 8xz + \frac{2y\lambda}{b^2} = 0$$

(x) by $y/2$ $\Rightarrow \frac{\cancel{8}^{4}xyz}{2} + \frac{\cancel{2}y^2\lambda}{\cancel{2}b^2} = 0$

$$4xyz + \frac{y^2\lambda}{b^2} = 0$$

$$4xyz = \frac{-y^2\lambda}{b^2}$$

$$\boxed{\frac{4xyz}{-\lambda} = \frac{y^2}{b^2}} \rightarrow (2)$$

$$\frac{\partial F}{\partial z} = 8xy + \frac{2z\lambda}{c^2}$$

$$\frac{\partial F}{\partial z} = 0$$

$$\Rightarrow 8xy + \frac{2z\lambda}{c^2} = 0$$

(x) by $\frac{z}{2}$ $\Rightarrow \frac{\cancel{8}^{4}xyz}{\cancel{2}} + \frac{\cancel{2}z^2\lambda}{\cancel{2}c^2}$

$$4xyz + \frac{z^2\lambda}{c^2}$$

$$4xyz = \frac{-z^2\lambda}{c^2}$$

$$\boxed{\frac{4xyz}{-\lambda} = \frac{z^2}{c^2}} \rightarrow (3)$$

from 1, 2, 3 we get

$$\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2} \longrightarrow 4$$

Equ (B) $\Rightarrow$ $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$$\frac{x^2}{a^2} + \frac{x^2}{a^2} + \frac{x^2}{a^2} = 1$$

$$\frac{3x^2}{a^2} = 1$$

$$x^2 = \frac{a^2}{3}$$

$$\boxed{x = \frac{a}{\sqrt{3}}}, \quad \boxed{y = \frac{b}{\sqrt{3}}}, \quad \boxed{z = \frac{c}{\sqrt{3}}}$$

∴ the Extremum Points $\left(a/\sqrt{3}, b/\sqrt{3}, c/\sqrt{3}\right)$

maximum volume, $V = 8xyz$

$$= 8\left(a/\sqrt{3}\right)\left(b/\sqrt{3}\right)\left(c/\sqrt{3}\right)$$

$$V = \frac{8abc}{3\sqrt{3}} \quad //$$

MA3151-MATRICES AND CALCULUS

UNIT-4

INTEGRAL CALCULUS

Mr.V.PRAKASH,M.Sc;M.Phil;B.Ed;
Assistant Professor
Department of Mathematics

UNIT-4 [INTEGRAL CALCULUS]

Definite Integrals:-

If 'f' is a function on $[a,b]$, then

$$\int_a^b f(x)\,dx = \lim_{n\to 0} \sum_{i=1}^{n} f(x_i)\,\Delta x, \quad \text{where } \Delta x = \frac{b-a}{n}$$

Here $\sum_{i=1}^{n} f(x_i)\,\Delta x$ is called Riemann sum.

Example-① Evaluate the Riemann sum for $f(x) = x^3 - 6x$, taking the sample points to be right end points and $a=0$; $b=3$ and $n=6$.

Soln:

Given $n=6$; $a=0$; $b=3$ and $f(x) = x^3 - 6x$

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{6} = 3/6 = 1/2 = 0.5$$

$x_0 = 0$, $x_1 = 0.5$, $x_2 = 1.0$, $x_3 = 1.5$, $x_4 = 2.0$, $x_5 = 2.5$, $x_6 = 3.0$

The Riemann sum is

$$R_6 = \sum_{i=1}^{6} f(x_i)\,\Delta x$$

$$= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x + f(x_6)\Delta x$$

$$= \Delta x\,[f(0.5) + f(1.0) + f(1.5) + f(2.0) + f(2.5) + f(3.0)]$$

$$= 0.5\,[-2.875 - 5 - 5.625 - 4 + 0.625 + 9]$$

$$= 0.5\,[-7.875]$$

$$R_6 = -3.9375$$

Example-2 Evaluate the Riemann sum for $\int_0^3 (x^3-6x)\,dx$.

Sol'n:

Given that $a=0$; $b=3$; & $f(x) = x^3-6x$

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = 3/n \text{ and } x_i = 3i/n$$

$$\therefore \int_0^3 (x^3-6x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right] 3/n$$

$$= \lim_{n\to\infty} 3/n \left[\sum_{i=1}^{n} \left(\frac{27i^3}{n^3} - \frac{18i}{n}\right)\right]$$

$$\cancel{= \lim_{n\to\infty} \left(\frac{3}{n}\right)\left[\frac{27}{n^3}\sum_{i=1}^{n}(i^3) - \frac{54}{n}\sum_{i=1}^{n} 1/n\right.}$$

$$= \lim_{n\to\infty} 3/n \left[\sum_{i=1}^{n} \frac{27i^3}{n^3} - \sum_{i=1}^{n} \frac{18i}{n}\right]$$

$$= \lim_{n\to\infty} \left\{ \frac{81}{n^4}\sum_{i=1}^{n} i^3 - \frac{54}{n^2}\sum_{i=1}^{n} i\right\}$$

$$= \lim_{n\to\infty} \left\{ \frac{81}{n^4}(1^3+2^3+\cdots+n^3) - \frac{54}{n^2}(1+2+\cdots+n)\right\}$$

$$= \lim_{n\to\infty} \left\{ \frac{81}{n^4}\left[\frac{n(n+1)}{2}\right]^2 - \frac{54}{n^2}\left[\frac{n(n+1)}{2}\right]\right\}$$

$$= \lim_{n\to\infty} \left\{ \frac{81}{n^4}\,\frac{n^2(n+1)^2}{4} - \frac{54}{n^2}\,\frac{n(n)(1+1/n)}{2}\right\}$$

$$= \lim_{n\to\infty} \left\{ \frac{81}{n^4}\,\frac{n^2(n^2)(1+1/n)^2}{4} - \frac{54}{\cancel{n^2}}\,\frac{\cancel{n^2}(1+1/n)}{2}\right\}$$

$$= \lim_{n\to\infty} \left\{ \frac{81}{\cancel{n^4}}\,\frac{\cancel{n^4}(1+1/n)^2}{4} - \frac{\overset{27}{\cancel{54}}(1+1/n)}{\cancel{2}}\right\}$$

$$= \lim_{n\to\infty} \left\{ \frac{81}{4}(1+1/n)^2 - 27(1+1/n) \right\}$$

W.K.T

$\left\{ 1/\infty = 0 \right\}$

$$= \frac{81}{4}(1+1/\infty)^2 - 27(1+1/\infty)$$

$$= \frac{81}{4}(1+0) - 27(1+0)$$

$$= \frac{81}{4} - 27 = \frac{81-108}{4}$$

$$R = \frac{-27}{4}$$

Example-(3) Prove that $\int_a^b x\,dx = \frac{b^2-a^2}{2}$.

Soln

Given that $f(x) = x$; and $\Delta x = \frac{b-a}{n}$, $x_i = a + \left(\frac{b-a}{n}\right)i$

$$\int_a^b x\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x)\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[a + \left(\frac{b-a}{n}\right)i\right]\left(\frac{b-a}{n}\right)$$

$$= \lim_{n\to\infty} \left(\frac{b-a}{n}\right) \sum_{i=1}^{n} a + \left(\frac{b-a}{n}\right)i$$

$$= \lim_{n\to\infty} \left\{ \frac{a(b-a)}{n} \sum_{i=1}^{n}(1) + \frac{(b-a)^2}{n^2} \sum_{i=1}^{n}(i) \right\}$$

$$= \lim_{n\to\infty} \left\{ a(b-a) + \frac{(b-a)^2}{n^2}(1+2+\cdots+n) \right\}$$

$$= \lim_{n\to\infty} \left\{ a(b-a) + \frac{(b-a)^2}{n^2}\,\frac{n(n+1)}{2} \right\}$$

$$= \lim_{n\to\infty} \left\{ a(b-a) + \frac{(b-a)^2}{\cancel{n^2}}\,\frac{\cancel{n^2}(1+1/n)}{2} \right\}$$

$$= \lim_{n\to\infty}\left\{ a(b-a) + \frac{(b-a)^2}{2}\left(1+\tfrac{1}{n}\right)\right\}$$

$$= a(b-a) + \frac{(b-a)^2}{2}\left(1+\tfrac{1}{\infty}\right) \qquad \left\{ \tfrac{1}{\infty} = 0 \right\}$$

$$= a(b-a) + \frac{(b-a)^2}{2}(1)$$

$$= (b-a)\left[a + \frac{(b-a)}{2}\right]$$

$$= (b-a)\left[\frac{2a+b-a}{2}\right]$$

$$= (b-a)\frac{(a+b)}{2} = \frac{(b-a)(b+a)}{2}$$

$$R = \frac{b^2-a^2}{2}$$

Example-(2) Evaluate $\int_0^3 (x^2-2x)\,dx$ by using Riemann sum by taking right end points as the sample points.

Soln: Given that $\int_0^3 (x^2-2x)\,dx$

Here, $a=0$; $b=3$, $\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = 3/n$ & $x_i = \frac{3i}{n}$

$$\int_0^3 (x^2-2x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n} f(3i/n)\,3/n$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(\frac{3i}{n}\right)^2 - 2\left(\frac{3i}{n}\right)\right](3/n)$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left[\frac{9}{n^2}(i)^2 - \frac{6}{n}i\right](3/n)$$

$$= \lim_{n\to\infty}(3/n)\sum_{i=1}^{n}\left[\frac{9}{n^2}i^2 - 6/n\, i\right]$$

$$= \lim_{n\to\infty} (3/n)\left\{ \sum_{i=1}^{n} 9/n^2\, i^2 - \sum_{i=1}^{n} 6/n\, i \right\}$$

$$= \lim_{n\to\infty} (3/n)\left\{ 9/n^2 \sum_{i=1}^{n} i^2 - 6/n \sum_{i=1}^{n} i \right\}$$

$$= \lim_{n\to\infty} (3/n)\left\{ 9/n^2 \left[1^2+2^2+\dots+n^2\right] - 6/n\left[1+2+\dots+n\right] \right\}$$

$$= \lim_{n\to\infty} (3/n)\left\{ 9/n^2 \frac{n(n+1)(2n+1)}{6} - 6/n \frac{n(n+1)}{2} \right\}$$

$$= \lim_{n\to\infty} \left\{ \frac{27}{n^3} \frac{n\cdot n(1+1/n)\, n(2+1/n)}{6} - 18/n^2 \frac{n\cdot n(1+1/n)}{2} \right\}$$

$$= \lim_{n\to\infty} \left\{ \frac{27}{\cancel{n^3}} \frac{\cancel{n^3}(1+1/n)(2+1/n)}{6} - \frac{18}{\cancel{n^2}} \cancel{n^2} \frac{(1+1/n)}{2} \right\}$$

$$= \lim_{n\to\infty} \left\{ \frac{27}{6}(1+1/n)(2+1/n) - \frac{\cancel{18}^{\,9}}{\cancel{2}}(1+1/n) \right\}$$

$$= \frac{27}{6} \lim_{n\to\infty} (1+1/n)(2+1/n) - 9 \lim_{n\to\infty} (1+1/n)$$

$$= \frac{27}{6} (1+1/\infty)(2+1/\infty) - 9\,(1+1/\infty)$$

$$= \frac{27}{\cancel{6}_3} (1)(\cancel{2}) - 9(1)$$

$$= \frac{27}{3} - 9$$

$$= 9 - 9$$

$$R = 0$$

Example-5 Find the Riemann sum for $f(x) = \sin x,\ 0 \le x \le 3\pi/2$ with six terms, taking the sample points to right end points.

Sol/n

Given that $f(x) = \sin x\ ,\ 0 \le x \le 3\pi/2$

Here, $a = 0,\ b = 3\pi/2\ \&\ \Delta x = \frac{b-a}{n} = \frac{3\pi/2 - 0}{6} = \frac{\cancel{3}\pi}{\cancel{12}_{4}}$

$n = 6$

$\boxed{\Delta x = \frac{\pi}{4}}$.

$x_1 = \pi/4 \quad x_4 = 4\pi/4$

$x_2 = 2\pi/4 \quad x_5 = 5\pi/4$

$x_3 = 3\pi/4 \quad x_6 = 6\pi/4$

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x$$

$$R_6 = \sum_{i=1}^{6} f(x_i)\Delta x$$

$$= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x + f(x_6)\Delta x$$

$$= \Delta x\left[f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) + f(x_6)\right]$$

$$= \pi/4\left[\sin(\pi/4) + \sin(2\pi/4) + \sin(3\pi/4) + \sin(4\pi/4) + \sin(5\pi/4) + \sin(6\pi/4)\right]$$

$$= \pi/4\left[\sin \pi/4 + \sin \pi/2 + \sin 3\pi/4 + \sin \pi + \sin 5\pi/4 + \sin 3\pi/2\right]$$

$$= \pi/4\left[\frac{1}{\sqrt{2}} + \cancel{1} + \cancel{\frac{1}{\sqrt{2}}} + 0 \cancel{- 1/\sqrt{2}} \cancel{- 1}\right]$$

$$= \pi/4\left[\frac{1}{\sqrt{2}}\right]$$

$$= \frac{\pi}{4\sqrt{2}}$$

$$\boxed{R_6 = 0.55536}$$ //

Chapter - 4.1

The Fundamental theorem of calculus - Part - 1

If f is continuous on $[a, b]$ then the function 'g' is defined by $g(x) = \int_a^x f(t)\,dt$, $a \le x \le b$, is continuous on $[a, b]$ and differentiable on (a, b) and $g'(x) = f(x)$.

The Fundamental theorem of calculus - Part - 2

If f is continuous on $[a, b]$ then $\int_a^b f(x)\,dx = F(b) - F(a)$ where F is any anti-derivative of f. (ie) a function s.t $F' = f$.

Example - ① Find the derivative of the function $g(x) = \int_0^x \sqrt{1+t^2}\,dt$.

Soln:

Since $f(t) = \sqrt{1+t^2}$ is continuous.

Given that $g(x) = \int_0^x \sqrt{1+t^2}\,dt$

W.K.T fundamental theorem of calculus part-1

$$g'(x) = \sqrt{1+x^2}$$

Example - ② Evaluate the integral $\int_1^3 e^x\,dx$.

Soln: Given that $\int_1^3 e^x\,dx = [e^x]_1^3$

$$= e^3 - e^1$$

Example-(3) Find $\frac{d}{dx}\left[\int_1^{x^4} \sec t\, dt\right]$.

Soln

Given that $\frac{d}{dx}\left[\int_1^{x^4} \sec t\, dt\right]$

Here we have to use the chain rule in conjunction with fundamental theorem of calculus.

Put $u = x^4 \Rightarrow \frac{du}{dx} = 4x^3$

then $\frac{d}{dx}\left[\int_1^{x^4} \sec t\, dt\right] = \frac{d}{dx}\left[\int_1^{u} \sec t\, dt\right]$

$$= \frac{d}{du}\left[\int_1^{u} \sec t\, dt\right]\frac{du}{dx}$$

$$= \sec u \cdot \frac{du}{dx}$$

$$= \sec u \cdot 4x^3$$

$$= \sec(x^4)\, 4x^3$$

$$= 4x^3 \sec(x^4)$$

Example-(21) Evaluate $\int_3^6 \frac{1}{x}\, dx$

soln:

Given that $\int_3^6 \frac{1}{x}\, dx = \left[\log x\right]_3^6$

$$= \log 6 - \log 3$$

$$= \log(6/3)$$

$$= \log(2)$$

Example-(5) Find the area under the Parabola $y = x^2$ from 0 to 1.

Soln: Given that $f(x) = y = x^2$, $0 \le x \le 1$

$$\therefore \int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \left[\tfrac{1}{3} - \tfrac{0}{3}\right]$$

$$= \tfrac{1}{3} - 0 = \tfrac{1}{3} \,//$$

Example-(6) What is wrong with the following calculation

$$\int_{-1}^{3} (1/x^2)\,dx = \left[\frac{x^{-1}}{-1}\right]_{-1}^{3} = -\tfrac{1}{3} - 1 = -4/3.$$

Soln: The calculation is wrong because the answer is negative but $f(x) = 1/x^2 \ge 0$. and it says that $\int_a^b f(x)\,dx \ge 0$ when $f(x) \ge 0$.

The fundamental theorem of calculus applies only to continuous functions. Here we cannot apply because $f(x) = 1/x^2$ is not continuous on $[-1, 3]$.

Here $f(x)$ has an infinite discontinuity at $x = 0$.

So. $\int_{-1}^{3} (1/x^2)\,dx$ does not exist.

Example-(7) What is wrong with the Equations

$$\int_{-2}^{1} x^{-4}\,dx = \left[\frac{x^{-3}}{-3}\right]_{-2}^{1} = -3/8.$$

soln:

The calculation is not correct, because the answer is negative but $f(x) = \frac{1}{x^4} \geq 0$ and by the property of integrals $\int_a^b f(x)\,dx \geq 0$, when $f(x) \geq 0$.

The fundamental theorem of calculus applies to continuous functions. It cannot be applied here because $f(x) = \frac{1}{x^4}$ is not continuous on $[-2, 1]$

$\therefore$ $f(x)$ has an infinite discontinuity at $x = 0$.

So, $\int_{-2}^{1} 1/x^4$ (or) $\int_{-2}^{1} x^{-4}$ does not exist

Example-(8) Evaluate $\int_1^4 (5 - 2t + 3t^2)\,dt$.

soln:

Given $\int_1^4 (5 - 2t + 3t^2)\,dt$

$$= \left[5t - \frac{\cancel{2}t^2}{\cancel{2}} + \frac{\cancel{3}t^3}{\cancel{3}}\right]_1^4 = \left[5t - t^2 + t^3\right]_1^4$$

$$= \left[5(4) - (4)^2 + (4)^3\right] - \left[5(1) - (1)^2 + (1)^3\right]$$

$$= [20 - 16 + 64] - [5 - \cancel{1} + \cancel{1}]$$

$$= 68 - 5$$

$$= 63$$

Example-(9) Evaluate $\int_0^1 x^{4/5}\,dx$

Soln:

Given $\int_0^1 x^{4/5}\,dx = \left[\dfrac{x^{4/5+1}}{4/5+1}\right]_0^1 = \left[\dfrac{x^{9/5}}{9/5}\right]_0^1$

$= \left[\dfrac{5(x)^{9/5}}{9}\right]_0^1 = \dfrac{5(1)^{9/5}}{9} - (0) = \dfrac{5(1)}{9}$

$= 5/9$

Example-(10) Evaluate $\int_1^2 \frac{3}{t^4}\,dt$.

Soln:

Given that $\int_1^2 \dfrac{3}{t^4}\,dt = \int_1^2 3t^{-4}\,dt = 3\int_1^2 t^{-4}\,dt$

$= 3\left[\dfrac{t^{-4+1}}{-4+1}\right]_1^2 = \cancel{3}\left[\dfrac{t^{-3}}{-\cancel{3}}\right]_1^2 = -\left[t^{-3}\right]_1^2$

$= -\left[\dfrac{1}{t^3}\right]_1^2 = -\left[\dfrac{1}{(2)^3} - \dfrac{1}{(1)^3}\right] = -\left[\dfrac{1}{8} - \dfrac{1}{1}\right]$

$= -\left[1/8 - 1\right] = -\left[\dfrac{1-8}{8}\right] = -\left[-7/8\right]$

$= 7/8$ //

Example-(11) Evaluate $\int_{-1}^2 (x^3-2x)\,dx$

Soln:

Given that $\int_{-1}^2 (x^3-2x)\,dx = \left[\dfrac{x^4}{4} - \dfrac{\cancel{2}x^2}{\cancel{2}}\right]_{-1}^2 = \left[\dfrac{x^4}{4} - x^2\right]_{-1}^2$

$= \left[\dfrac{(2)^4}{4} - (2)^2\right] - \left[\dfrac{(-1)^4}{4} - (-1)^2\right]$

$= \left[\dfrac{\cancel{16}^{\,4}}{\cancel{4}} - 4\right] - \left[1/4 - 1\right] = [4-4] - [1/4 - 1]$

$= 0 - \left(\dfrac{1-4}{4}\right) = -(-3/4)$

$= 3/4$.

INDEFINITE INTEGRALS :-

Example-① Evaluate $\int (10x^4 - 2\sec^2 x)\,dx$

Soln:

Given $\int (10x^4 - 2\sec^2 x)\,dx$

$$= 10\int x^4\,dx - 2\int \sec^2 x\,dx$$

$$= \cancel{10}^{2}\left(\frac{x^5}{\cancel{5}}\right) - 2\tan x + C$$

$$= 2x^5 - 2\tan x + C$$

Example-② Evaluate $\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$

Soln:

Given $\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$

$$= \int \frac{\cos\theta}{\sin\theta\,\sin\theta}\,d\theta = \int \frac{\cos\theta}{\sin\theta}\cdot\frac{1}{\sin\theta}\,d\theta$$

$$= \int \cot\theta\,\operatorname{cosec}\theta\,d\theta$$

$$= \int \operatorname{cosec}\theta\cot\theta\,d\theta = -\operatorname{cosec}\theta + c$$

Example-③ Evaluate $\int (\operatorname{cosec}^2 t - 2e^t)\,dt$

Soln:

Given $\int (\operatorname{cosec}^2 t - 2e^t)\,dt$

$$= \int \operatorname{cosec}^2 t\,dt - 2\int e^t\,dt$$

$$= -\cot t - 2e^t + c$$

Example - (4) Evaluate $\int \sec t\,(\sec t + \tan t)\,dt$

Soln:

Given $\int \sec t\,(\sec t + \tan t)\,dt$

$$= \int (\sec^2 t + \sec t \tan t)\,dt$$

$$= \int \sec^2 t\,dt + \int \sec t \tan t\,dt$$

$$= \tan t + \sec t + C \,//$$

Example - (5) Evaluate $\int \frac{\sin 2x}{\sin x}\,dx.$

Soln:

Given $\int \frac{\sin 2x}{\sin x}\,dx$

$$= \int \frac{2\cancel{\sin x}\cos x}{\cancel{\sin x}}\,dx \qquad \{\text{W.K.T. } \sin 2\theta = 2\cos\theta\sin\theta$$

$$= \int 2\cos x\,dx$$

$$= 2\int \cos x\,dx$$

$$= 2\sin x + C \,//$$

Example - (6) Evaluate $\int \frac{3}{x^2+1}\,dx$

Soln:

Given $\int \frac{3}{x^2+1}\,dx \qquad \left\{\text{W.K.T } \int \frac{dx}{x^2+a^2} = \tan^{-1}(x/a)\right.$

$$= 3\int \frac{dx}{x^2+1}$$

$$= 3\tan^{-1}(x/1)$$

$$= 3\tan^{-1}x \,//$$

Example-7 Evaluate $\int\left(\sqrt{x^3}+\sqrt[3]{x^2}\right)dx$

Soln- Given $\int\left(\sqrt{x^3}+\sqrt[3]{x^2}\right)dx$

$$=\int\left[(x^3)^{1/2}+(x^2)^{1/3}\right]dx$$

$$=\int\left(x^{3/2}+x^{2/3}\right)dx$$

$$=\left[\frac{x^{3/2+1}}{3/2+1}+\frac{x^{2/3+1}}{2/3+1}\right]dx$$

$\left\{\begin{array}{l}\text{W.K.T}\\ 3/2+1=\frac{3+2}{2}=5/2\\ 2/3+1=\frac{2+3}{3}=5/3\end{array}\right.$

$$=\left[\frac{x^{5/2}}{5/2}+\frac{x^{5/3}}{5/3}\right]$$

$$=2/5\,x^{5/2}+3/5\,x^{5/3}$$

Example-8 Evaluate $\int(1+\tan^2\theta)\,d\theta$

Soln- Given $\int(1+\tan^2\theta)\,d\theta$

$$=\int(\sec^2\theta)\,d\theta$$

$$=\tan\theta+c$$

Example-9 Evaluate $\int(\sin x+\sinh x)\,dx$

Soln- Given $\int(\sin x+\sinh x)\,dx$

$$=-\cos x+\cosh x+c$$

Example - (10) Evaluate $\int \frac{1}{\sin^2 x \cos^2 x} dx$.

Soln:

Given $\int \frac{1}{\sin^2 x \cos^2 x} dx$

$$= \int \frac{\cos^2 x + \sin^2 x}{\sin^2 x \cos^2 x} dx = \int \left(\frac{\cancel{\cos^2 x}}{\sin^2 x \cancel{\cos^2 x}} + \frac{\cancel{\sin^2 x}}{\cancel{\sin^2 x} \cos^2 x} \right) dx$$

$$= \int \left(\frac{1}{\sin^2 x} + \frac{1}{\cos^2 x} \right) dx$$

$$= \int (\sec^2 x + \operatorname{cosec}^2 x) dx$$

$$= \tan x - \cot x + C$$

Example - (11) Evaluate $\int \frac{1}{1+\sin x} dx$

Soln:

Given $\int \frac{dx}{1+\sin x}$

$$= \int \frac{1}{1+\sin x} \times \frac{1-\sin x}{1-\sin x} dx$$

$$= \int \frac{1-\sin x}{(1)^2 - \sin^2 x} dx = \int \frac{1-\sin x}{1-\sin^2 x} dx$$

$$= \int \frac{1-\sin x}{\cos^2 x} dx = \int \left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x} \right) dx$$

$$= \int \left(\sec^2 x - \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} \right) dx$$

$$= \int (\sec^2 x - \tan x \sec x) dx$$

$$= \tan x - \sec x + C$$

Properties:-

$$\Rightarrow \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$\Rightarrow \int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$$

$$\Rightarrow \int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

$$\Rightarrow \int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

$$\Rightarrow \int_0^a f(x)\,dx = \int_a^b f(a-x)\,dx.$$

CHAPTER-42

METHODS OF INTEGRATION

1. substitution Rule
2. Integration by Parts
3. Integration by method of Partial fractions
4. successive Reduction method.

I- substitution method

Example-1 Evaluate $\int \frac{x^3}{\sqrt{1-x^8}}\,dx$

Sol^n:

Given that $\int \frac{x^3}{\sqrt{1-x^8}}\,dx$.

Put $u = x^4 \Rightarrow du = 4x^3\,dx$ | $x^4 = u$

$\frac{du}{4} = x^3\,dx$ | $x^8 = (u)^2$

$$\Rightarrow \int \frac{x^3\,dx}{\sqrt{1-x^8}} = \int \frac{1}{\sqrt{1-u^2}}\,\frac{du}{4} = \frac{1}{4}\int \frac{du}{\sqrt{1-u^2}}$$

$$= \tfrac{1}{4}\sin^{-1}(u) + C$$

$$\underline{I} = \tfrac{1}{4}\sin^{-1}(x^4) + C$$

$\left\{ \begin{array}{l} \text{W.K.T} \\ \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}(x/a) \end{array} \right.$

Example-2 Evaluate $\int \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx$

Sol^n:

Given that $\int \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx$

Put $\sin^{-1}x = t$

$\frac{1}{\sqrt{1-x^2}}\,dx = dt$

$$\Rightarrow \int \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx = \int t\,dt$$

$$= t^2/2 + c$$

$$I = \frac{(\sin^{-1}x)^2}{2} + c$$

Example-(3) Evaluate $\int \frac{\cos\theta}{\sin^3\theta}\,d\theta$. by method of substitution.

Soln:- Given that $\int \frac{\cos\theta}{\sin^3\theta}\,d\theta$

Put, $t = \sin\theta \Rightarrow dt = \cos\theta\,d\theta$

$$\Rightarrow \int \frac{\cos\theta\,d\theta}{\sin^3\theta} = \int \frac{dt}{t^3} = \int t^{-3}\,dt = \left[\frac{t^{-3+1}}{-3+1}\right]$$

$$= \frac{t^{-2}}{-2} = \frac{-1}{2t^2}$$

$$= -\tfrac{1}{2}\,\frac{1}{t^2} = -\tfrac{1}{2}\,\frac{1}{\sin^2\theta} = \frac{-1}{2\sin^2\theta}$$

$$I = -\tfrac{1}{2}\operatorname{Cosec}^2\theta + c$$

Example-(4) Evaluate $\int x^3\cos(x^4+2)\,dx$

Soln:- Given that $\int x^3\cos(x^4+2)\,dx$

Put $x^4+2 = u \Rightarrow 4x^3\,dx = du$

$$x^3\,dx = \frac{du}{4}$$

$$\Rightarrow \int \cos(x^4+2)\,x^3\,dx = \int \cos(u)\,\frac{du}{4} = \tfrac{1}{4}\int \cos u\,du$$

$$= \tfrac{1}{4}\sin u$$

$$I = \tfrac{1}{4}\sin(x^4+2) + C$$

Example-5 Evaluate $\int \frac{x}{\sqrt{1-4x^2}}\,dx$

Soln:-

Given that $\int \frac{x\,dx}{\sqrt{1-4x^2}}$

Put, $1-4x^2 = u \Rightarrow -8x\,dx = du$

$$x\,dx = \frac{du}{-8}$$

$$\Rightarrow \int \frac{x\,dx}{\sqrt{1-4x^2}} = \int \frac{(du/-8)}{\sqrt{u}} = -1/8 \int \frac{du}{\sqrt{u}}$$

$$= -1/8 \int \frac{du}{(u)^{1/2}} = -1/8 \int u^{-1/2}\,du$$

$$= -1/8 \left[\frac{u^{-1/2+1}}{-1/2+1}\right] = -1/8 \left[\frac{u^{1/2}}{1/2}\right]$$

$$= -1/8 \left[2u^{1/2}\right] = -1/4\, u^{1/2} = -1/4 \sqrt{u}$$

$$I = -1/4 \sqrt{1-4x^2} + C$$

Example-6 Evaluate $\int \sqrt{1+x^2}\, x^5\,dx$.

Soln:-

Given that $\int \sqrt{1+x^2}\cdot x^5\,dx$

Put, $1+x^2 = u \Rightarrow 2x\,dx = du$

$$x\,dx = du/2$$

$u = 1+x^2$

$x^2 = 1-u$

$x^4 = (1-u)^2$

$$\Rightarrow \int \sqrt{1+x^2}\, x^5\,dx = \int \sqrt{1+x^2}\cdot x^4\cdot x\,dx$$

$$= \int \sqrt{u}\cdot (1-u)^2\cdot du/2 = 1/2 \int \sqrt{u}\,(1-u)^2\,du$$

$$= 1/2 \int u^{1/2}\,(1+u^2-2u)\,du$$

$$= 1/2 \int (u^{1/2} + u^{2+1/2} - 2u^{1+1/2})\,du$$

$$= 1/2 \int (u^{1/2} + u^{5/2} - 2u^{3/2})\,du$$

$$= \frac{1}{2}\int \left(u^{1/2} + u^{5/2} - 2u^{3/2}\right) du$$

$$= \frac{1}{2}\left[\frac{u^{1/2+1}}{1/2+1} + \frac{u^{5/2+1}}{5/2+1} - \frac{2u^{3/2+1}}{3/2+1}\right]$$

$$= \frac{1}{2}\left[\frac{u^{3/2}}{3/2} + \frac{u^{7/2}}{7/2} - \frac{2u^{5/2}}{5/2}\right]$$

$$= \frac{1}{2}\left[\frac{2}{3}u^{3/2} + \frac{2}{7}u^{7/2} - \frac{4}{5}u^{5/2}\right]$$

$$= \frac{1}{3}u^{3/2} + \frac{1}{7}u^{7/2} - \frac{2}{5}u^{5/2}$$

$$I = \frac{1}{3}(1+x^2)^{3/2} + \frac{1}{7}(1+x^2)^{7/2} - \frac{2}{5}(1+x^2)^{5/2}$$

Example-(7) Evaluate $\int \sqrt{2x+1}\, dx$

Soln Given that $\int \sqrt{2x+1}\, dx$

Put $u = 2x+1 \Rightarrow du = 2dx$

$$\frac{du}{2} = dx$$

$$\Rightarrow \int \sqrt{2x+1}\, dx = \int \sqrt{u} \cdot \frac{1}{2} du = \int (u^{1/2}) \frac{1}{2} du$$

$$= \frac{1}{2}\int u^{1/2} du = \frac{1}{2}\left[\frac{u^{1/2+1}}{1/2+1}\right]$$

$$= \frac{1}{2}\left[\frac{u^{3/2}}{3/2}\right] = \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]$$

$$= \frac{1}{3}u^{3/2} + c$$

$$I = \frac{1}{3}(2x+1)^{3/2} + c$$

Example-(8) Evaluate $\int_1^2 \frac{dx}{(3-5x)^2}$.

Soln:

Given that $\int_1^2 \frac{dx}{(3-5x)^2}$

Put, $(3-5x) = u \Rightarrow -5\,dx = du$

$$dx = \frac{du}{-5}$$

ie, $u = 3-5x$

Put $x=1 \Rightarrow u = 3-5(1) = 3-5 = -2 \Rightarrow \boxed{u=-2}$

$x=2 \Rightarrow u = 3-5(2) = 3-10 = -7 \Rightarrow \boxed{u=-7}$

$$\Rightarrow \int_1^2 \frac{dx}{(3-5x)^2} = \int_{-7}^{-2} \frac{(du/-5)}{u^2} = -1/5 \int_{-7}^{-2} \frac{du}{u^2}$$

$$= -1/5 \int_{-7}^{-2} \frac{du}{u^2} = -1/5 \int_{-2}^{-7} \left(\frac{1}{u^2}\right) du$$

$$= -1/5 \left[-1/u\right]_{-2}^{-7} = 1/5 \left[\frac{1}{u}\right]_{-2}^{-7}$$

$$= 1/5 \left[\frac{1}{-7} - \frac{1}{-2}\right] = 1/5 \left[-1/7 + 1/2\right]$$

$$= 1/5 \left[\frac{-2+7}{14}\right] = 1/\cancel{5} \left[\cancel{5}/14\right]$$

$$\boxed{I = 1/14}$$

Example - 9 Evaluate $\int_{-2}^{2} (x^6+1)\,dx$.

Soln: Given $\int_{-2}^{2} (x^6+1)\,dx$

$$\left\{ \text{W.K.T} \quad \int_{-\infty}^{\infty} f(x)\,dx = 2\int_{0}^{\infty} f(x)\,dx \right.$$

$$= 2\int_{0}^{2} (x^6+1)\,dx$$

$$= 2\left[\frac{x^7}{7}+x\right]_0^2$$

$$= 2\left[\frac{(2)^7}{7}+2\right] - [0]$$

$$= 2\left[\frac{128}{7}+2\right] = 2\left[\frac{128+14}{7}\right] = 2\left[\frac{142}{7}\right]$$

$$I = \frac{284}{7}$$

II - Integration By Parts method

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Example-① Evaluate $\int x \sin x\,dx$ by using integration by parts.

Soln: Given that $\int x \sin x\,dx$.

Let $u = x$ | $dv = \sin x\,dx$ | $u \to$ diff

$du = dx$ | $v = -\cos x$ | $dv \to$ Integ

$$\int u\,dv = uv - \int v\,du$$

$$\int x \sin x\,dx = x(-\cos x) - \int(-\cos x)\,dx$$

$$= -x\cos x + \int \cos x\,dx$$

$$= -x\cos x + \sin x + C$$

Example-② Evaluate $\int \log x\,dx$.

Soln: Given that $\int \log x\,dx$

Let $u = \log x$ | $dv = dx$

$du = \frac{1}{x}dx$ | $v = x$

W.K.T $\int u\,dv = uv - \int v\,du$

$$\int \log x\,dx = \log x\,(x) - \int \cancel{x}\,\frac{1}{\cancel{x}}\,dx$$

$$= x\log x - \int dx$$

$$= x\log x - x + C$$

Example-(3) Find $\int t^2 e^t \, dt$ by using integration by parts.

Soln. Given that $\int t^2 e^t \, dt$.

Here, $u = t^2$ | $dv = e^t dt$

$du = 2t\,dt$ | $v = e^t$

W.K.T $\int u\,dv = uv - \int v\,du$

$$\int t^2 e^t \, dt = t^2 (e^t) - \int e^t \, 2t\,dt$$

$$= e^t t^2 - 2\int t\, e^t \, dt \longrightarrow (1)$$

Again apply integration by parts on second terms of R.H.S

Now Consider, $\int t\, e^t \, dt$

let $u = t$ | $dv = e^t dt$

$du = dt$ | $v = e^t$

W.K.T $\int u\,dv = uv - \int v\,du$

$$\int t e^t \, dt = t(e^t) - \int e^t \, dt$$

$$= t e^t - e^t + c \longrightarrow (2)$$

use (2) in (1)

$$\int t^2 e^t \, dt = e^t t^2 - 2\int t e^t \, dt$$

$$= t^2 e^t - 2[t e^t - e^t] + c$$

$$= t^2 e^t - 2t e^t + 2e^t + c$$

Example - (4) Evaluate $\int e^x \sin x \, dx$. by using integration by Parts.

Soln:

Given that $\int e^x \sin x \, dx$

Here, $u = e^x$ | $dv = \sin x \, dx$

$du = e^x dx$ | $v = -\cos x$

$$\therefore \int u \, dv = uv - \int v \, du$$

$$\int e^x \sin x \, dx = e^x(-\cos x) - \int (-\cos x) e^x dx$$

$$= -e^x \cos x + \underbrace{\int e^x \cos x \, dx}$$

Again apply integration by parts on second term of R.H.S

Here $u = e^x$ | $dv = \cos x \, dx$

$du = e^x dx$ | $v = \sin x$

$$\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - \int \sin x \, e^x dx$$

$$= -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx$$

Adding $\int e^x \sin x \, dx$ on both side,

$$\int e^x \sin x \, dx + \int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - \cancel{\int e^x \sin x dx} + \cancel{\int e^x \sin x dx}$$

$$2 \int e^x \sin x \, dx = e^x \sin x - e^x \cos x$$

$$\int e^x \sin x \, dx = \frac{1}{2}\left[e^x \sin x - e^x \cos x\right]$$

$$= \frac{e^x}{2}\left[\sin x - \cos x\right] + C$$

Example - 5 Evaluate $\int \left(\frac{\cancel{\log x}}{x}\right)^2 dx$ by using Integration by parts.

Soln Let given that $\int \left(\frac{\log x}{x}\right)^2 dx$

$$= \int \frac{(\log x)^2}{x^2} dx$$

Here $u = (\log x)^2$ | $dv = \frac{1}{x^2} dx$

$du = \frac{2\log x}{x} dx$ | $v = -\frac{1}{x}$

$\left\{ \frac{1}{x^2} = x^{-2} \quad \int x^{-2} dx = \frac{x^{-2+1}}{-2+1} = \frac{x^{-1}}{-1} = \frac{1}{-x} \right.$

$$\int u\,dv = uv - \int v\,du$$

$$\int \frac{(\log x)^2}{x^2} dx = (\log x)^2 (-1/x) - \int (-1/x)\left(2\frac{\log x}{x}\right) dx$$

$$= \frac{-(\log x)^2}{x} + \int \frac{2\log x}{x^2} dx$$

$$= -\frac{(\log x)^2}{x} + 2\int \frac{\log x}{x^2} dx$$

Again using Integration by parts

Here, $u = \log x$ | $dv = \frac{1}{x^2} dx$

$du = \frac{1}{x} dx$ | $v = -\frac{1}{x}$ $\quad \int u\,dv = uv - \int v\,du$

$$= -\frac{(\log x)^2}{x} + 2\left\{ \log x(-1/x) - \int (-1/x) \frac{1}{x} dx \right.$$

$$= -\frac{(\log x)^2}{x} + 2\left\{ -\frac{1}{x}\log x + \int \frac{1}{x^2} dx \right\}$$

$$= -\frac{(\log x)^2}{x} + 2\left\{ -\frac{\log x}{x} - \frac{1}{x} \right\} + C$$

$$\int \frac{(\log x)^2}{x^2} = -\frac{(\log x)^2}{x} - 2\frac{\log x}{x} - 2/x + C \;//$$

Example-6 Evaluate $\int e^{ax}\cos bx\,dx$ by using Integration by parts.

Soln:-

Let $I = \int e^{ax}\cos bx\,dx \longrightarrow (1)$

where a & b are constant and $a \neq 0$ and $b \neq 0$.

Here. $u = e^{ax}$, $du = a e^{ax}\,dx$ | $dv = \cos bx\,dx$, $v = \frac{\sin bx}{b}$

W.K.T $\int u\,dv = uv - \int v\,du$.

$$I = e^{ax}\frac{\sin bx}{b} - \int \frac{\sin bx}{b} a e^{ax}\,dx$$

$$= \frac{1}{b} e^{ax}\sin bx - \frac{a}{b}\int e^{ax}\sin bx\,dx$$

Again using Integration by parts.

Here. $u = e^{ax}$, $du = a e^{ax}\,dx$ | $dv = \sin b\,dx$, $V = \frac{-\cos bx}{b}$ | $\int u\,dv = uv - \int v\,du$

$$I = \frac{1}{b} e^{ax}\sin bx - \frac{a}{b}\left\{ e^{ax}\left(-\frac{\cos bx}{b}\right) - \int -\frac{\cos bx}{b} a e^{ax}\,dx\right\}$$

$$= \frac{1}{b} e^{ax}\sin bx + \frac{a}{b^2} e^{ax}\cos bx - \frac{a^2}{b^2}\int e^{ax}\cos bx\,dx$$

{using (1)}

$$I = \frac{1}{b} e^{ax}\sin bx + \frac{a}{b^2} e^{ax}\cos bx - \frac{a^2}{b^2} I$$

$$I + \frac{a^2}{b^2} I = \frac{1}{b} e^{ax}\sin bx + \frac{a}{b^2} e^{ax}\cos bx$$

$$I\left(1 + \frac{a^2}{b^2}\right) = \frac{e^{ax}}{b}\left[\sin bx + \frac{a}{b}\cos bx\right]$$

$$I\left(\frac{b^2 + a^2}{b^2}\right) = \frac{e^{ax}}{b}\left[\sin bx + \frac{a}{b}\cos bx\right].$$

$$I\left(\frac{a^2+b^2}{b^2}\right) = \frac{e^{ax}}{b}\left[a/b \cos bx + \sin bx\right]$$

$$I = \left(\frac{b^2}{a^2+b^2}\right)\frac{e^{ax}}{b}\left[a/b \cos bx + \sin bx\right]$$

$$= \left(\frac{b}{a^2+b^2}\right) e^{ax}\left[a/b \cos bx + \sin bx\right]$$

$$= \frac{e^{ax}}{a^2+b^2}\left[\cancel{ab/b} \cos bx + b\sin bx\right]$$

$$\underline{I} = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right] //$$

Example-7 Evaluate $\int e^{-ax} \sin bx$, by using Integration by parts.

Soln:

Let $I = \int e^{-ax} \sin bx \, dx \longrightarrow (1)$

where a & b are constants and $a \neq 0$, $b \neq 0$.

Here, $u = e^{-ax}$ | $dv = \sin bx$

$du = -a e^{-ax} dx$ | $v = -\frac{\cos bx}{b}$

W.K.T $\int u\,dv = uv - \int v\,du$

$$I = e^{-ax}\left(-\frac{\cos bx}{b}\right) - \int\left(-\frac{\cos bx}{b}\right)\left(-a e^{-ax}\right)dx$$

$$= -1/b\, e^{-ax}\cos bx - a/b\int e^{-ax}\cos bx\, dx$$

$$= -1/b\, e^{-ax}\cos bx - a/b\int e^{-ax}\cos bx\, dx.$$

$\{\int u\,dv = uv - \int v\,du\}$

Again using Integration parts.

Here $u = e^{-ax}$ | $dv = \cos bx \, dx$

$du = -a e^{-ax} dx$ | $v = \frac{\sin bx}{b}$

$$I = -\frac{1}{b} e^{-ax} \cos bx - \frac{a}{b} \left\{ e^{-ax} \frac{\sin bx}{b} - \int \frac{\sin bx}{b} (-a e^{-ax}) dx \right.$$

$$= -\frac{1}{b} e^{-ax} \cos bx - \frac{a}{b^2} e^{-ax} \sin bx - \frac{a^2}{b^2} \int e^{-ax} \sin bx \, dx$$

{using (1)}

$$I = -\frac{1}{b} e^{-ax} \cos bx - \frac{a}{b^2} e^{-ax} \sin bx - \frac{a^2}{b^2} I$$

$$I + \frac{a^2}{b^2} I = -\frac{1}{b} e^{-ax} \cos bx - \frac{a}{b^2} e^{-ax} \sin bx$$

$$I\left(1 + \frac{a^2}{b^2}\right) = \frac{e^{-ax}}{b} \left(-\cos bx - \frac{a}{b} \sin bx\right)$$

$$I\left(\frac{b^2 + a^2}{b^2}\right) = \frac{e^{-ax}}{b} \cdot \left(-\cos bx - \frac{a}{b} \sin bx\right)$$

$$I = \left(\frac{b^{\cancel{2}}}{a^2 + b^2}\right) \frac{e^{-ax}}{\cancel{b}} \left(-\cos bx - \frac{a}{b} \sin bx\right)$$

$$= \left(\frac{e^{-ax}}{a^2 + b^2}\right) b \left(-\cos bx - \frac{a}{b} \sin bx\right)$$

$$= \frac{e^{-ax}}{a^2 + b^2} \left(-b \cos bx - \frac{a\cancel{b}}{\cancel{b}} \sin bx\right)$$

$$I = \frac{e^{-ax}}{a^2 + b^2} \left[-b \cos bx - a \sin bx\right]$$

Example-8 Evaluate $\int \sin^n x\,dx$, by using integration by parts.

Soln:

let us consider $I_n = \int \sin^n x\,dx \longrightarrow (1)$

$$I_n = \int \sin^{n-1}x \cdot \sin x\,dx$$

Here, $u = \sin^{n-1}x$ | $dv = \sin x\,dx$

$du = (n-1)\sin^{n-2}x \cos x\,dx$ | $V = -\cos x$

W.K.T $\int u\,dv = uv - \int v\,du$

$$I_n = \sin^{n-1}x(-\cos x) - \int(-\cos x)(n-1)\sin^{n-2}\cos x\,dx$$

$$= -\cos x \sin^{n-1}x + (n-1)\int \cos^2 x \sin^{n-2}x\,dx$$

$$= -\cos x \sin^{n-1}x + (n-1)\int (1-\sin^2 x)\sin^{n-2}x\,dx$$

$$= -\cos x \sin^{n-1}x + (n-1)\int\left(\sin^{n-2}x - \sin^{n-2+2}x\right)dx$$

$$= -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x\,dx - (n-1)\int \sin^n x\,dx$$

{using (1)}

$$I_n = -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x\,dx - (n-1)I_n$$

$$I_n + (n-1)I_n = -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x\,dx$$

$$I_n(\cancel{1}+n-\cancel{1}) = -\cos x \sin^{n-1}x + (n-1)I_{n-2}$$

$$nI_n = -\cos x \sin^{n-1}x + (n-1)I_{n-2}$$

$$I_n = \frac{-\cos x \sin^{n-1}x}{n} + \frac{(n-1)}{n}I_{n-2}.$$

Example-9 Evaluate $\int \cos^n x \, dx$, by using Integration by Parts.

Soln:

Let us consider $I_n = \int \cos^n x \, dx \longrightarrow ①$

$$I_n = \int \cos^{n-1} x \cos x \, dx$$

Here, $u = \cos^{n-1} x$ | $dv = \cos x \, dx$

$du = (n-1) \cos^{n-2} x (-\sin x) dx$ | $v = \sin x$

$du = -(n-1) \cos^{n-2} x \sin x \, dx$

W.K.T $\int u \, dv = uv - \int v \, du$

$$I_n = \cos^{n-1} x \sin x - \int \sin x \left[-(n-1) \cos^{n-2} x \sin x\right] dx$$

$$= \cos^{n-1} x \sin x + \int (n-1) \cos^{n-2} x \sin^2 x \, dx$$

$$= \cos^{n-1} x \sin x + \int \sin^2 x (n-1) \cos^{n-2} x \, dx$$

$$= \cos^{n-1} x \sin x + (n-1) \int (1 - \cos^2 x) \cos^{n-2} x \, dx$$

$$= \cos^{n-1} x \sin x + (n-1) \int (\cos^{n-2} x - \cos^{n-2+2} x) dx$$

$$= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \, dx - (n-1) \int \cos^n x \, dx$$

{using Eqn ①}

$$I_n = \cos^{n-1} x \sin x + (n-1) I_{n-2} - (n-1) I_n$$

$$I_n + (n-1) I_n = \cos^{n-1} x \sin x + (n-1) I_{n-2}$$

$$I_n [\cancel{1} + n - \cancel{1}] = \cos^{n-1} x \sin x + (n-1) I_{n-2}$$

$$n I_n = \cos^{n-1} x \sin x + (n-1) I_{n-2}$$

$$I_n = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} I_{n-2} \quad //$$

Example-10 Evaluate $\int_0^{\pi/2} \sin^n x\,dx$.

Soln:

Given that $\int_0^{\pi/2} \sin^n x\,dx$

W.K.T $\int \sin^n x\,dx = \frac{-\cos x \sin^{n-1} x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx$

$$\therefore \int_0^{\pi/2} \sin^n x\,dx = \left[\frac{-\cos x \sin^{n-1} x}{n}\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$$

$$= \left[\frac{-\cos \pi/2 \sin^{n-1} \pi/2}{n} - \frac{-\cos(0) \sin^{n-1}(0)}{n}\right] + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$$

$$= \left[\frac{-(0)(1)}{n} + \frac{(1)(0)}{n}\right] + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$$

$$= 0 + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$$

$$= \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$$

{Here the first term vanishes for both upper & lower limits}

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\int_0^{\pi/2} \sin^{n-4} x\,dx$$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\int_0^{\pi/2} \sin^{n-6} x\,dx$$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdot\frac{n-7}{n-6}\cdots\cdots I$$

If n is even, then,

$$I = \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \pi/2 - 0$$

$$\boxed{I = \pi/2}$$

If n is odd, then $I = \int_0^{\pi/2} \sin x \, dx$

$$= \left[-\cos x\right]_0^{\pi/2} = -\left[\cos \pi/2 - \cos(0)\right]$$

$$= -[0-1] = 1 \Rightarrow \boxed{I = 1}$$

$$\therefore \int_0^{\pi/2} \sin^n x \, dx = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{1}{2} \, \frac{\pi}{2}, & n \text{ is Even} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{2}{3}, & n \text{ is odd} \end{cases}$$

__Example-__ 11. Evaluate $\int \tan^{-1} x \, dx$. Also find $\int_0^1 \tan^{-1} x \, dx$

__Soln__ Given that $\int \tan^{-1} x \, dx$.

Here $u = \tan^{-1} x$ | $dv = dx$

$du = \frac{1}{1+x^2} dx$ | $v = x$

W.K.T $\int u \, dv = uv - \int v \, du$

$$\int \tan^{-1} x \, dx = \tan^{-1} x (x) - \int x \frac{1}{1+x^2} dx$$

$$I = x \tan^{-1} x - \int \frac{x \, dx}{1+x^2}$$

Put $t = 1+x^2$

$dt = 2x \, dx$

$\frac{dt}{2} = x \, dx$

$$I = x \tan^{-1} x - \int \frac{dt/2}{t}$$

$$I = x\tan^{-1}x - \frac{1}{2}\int \frac{dt}{t}$$

$$= x\tan^{-1}x - \frac{1}{2}\log t$$

$$I = x\tan^{-1}x - \frac{1}{2}\log(1+x^2)$$

Also we find,

$$\int_0^1 \tan^{-1}x\,dx = \left[x\tan^{-1}x - \frac{1}{2}\log(1+x^2)\right]_0^1$$

$$= \left[(1)\tan^{-1}(1) - \frac{1}{2}\log(1+1)\right] - \left[0 - \frac{1}{2}\log(1+0)\right]$$

$$= \left[\tan^{-1}(1) - \frac{1}{2}\log(2)\right] - \left[-\frac{1}{2}\log(1)\right]$$

$$= \pi/4 - \frac{1}{2}\log(2) - (0)$$

$$= \pi/4 - \frac{1}{2}\log(2) \quad //$$

III Trigonometric Integrals

① Evaluate $\int \cos^3 x\, dx$

Soln:

Given that $\int \cos^3 x\, dx$

Let $u = \sin x \Rightarrow du = \cos x\, dx$

W.k.T $\cos^2 x + \sin^2 x = 1 \Rightarrow \cos^2 x = 1 - \sin^2 x$

$$\int \cos^3 x\, dx = \int \cos^2 x \cos x\, dx$$

$$= \int (1 - \sin^2 x) \cos x\, dx$$

$$= \int (1 - u^2)\, du = \left[u - \frac{u^3}{3}\right] + c$$

$$I = \sin x - \frac{\sin^3 x}{3} + c$$

② Evaluate $\int \sin^5 x \cos^2 x\, dx$

Soln:

Given that $\int \sin^5 x \cos^2 x\, dx$

Let $\sin^5 x \cos^2 x = (\sin^2 x)^2 \sin x \cos^2 x$

$$= (1 - \cos^2 x)^2 \cos^2 x \sin x$$

Let us consider, $u = \cos x \Rightarrow du = -\sin x\, dx$

$$\int \sin^5 x \cos^2 x\, dx = \int (1 - \cos^2 x)^2 \cos^2 x \sin x\, dx$$

$$= \int (1 - u^2)^2 u^2 (-du)$$

$$= -\int (1 + u^4 - 2u^2) u^2\, du$$

$$= -\int (u^2 + u^6 - 2u^4)\, du$$

$$= -\int (u^2 + u^6 - 2u^4)\, du$$

$$= -\left[\frac{u^3}{3} + \frac{u^7}{7} - \frac{2u^5}{5}\right] + c$$

$$= -\frac{u^3}{3} - \frac{u^7}{7} + \frac{2u^5}{5} + c$$

$$I = -\frac{\cos^3 x}{3} - \frac{\cos^7 x}{7} + \frac{2\cos^5 x}{5} + c$$

(3) Evaluate: $\int \tan^3 x\, dx$.

Solⁿ: $\int \tan^3 x\, dx = \int \tan^2 x \tan x\, dx$

$$= \int (\sec^2 x - 1) \tan x\, dx$$

$$= \int (\sec^2 x \tan x - \tan x)\, dx$$

$$= \underbrace{\int \sec^2 x \tan x\, dx}_{} - \int \tan x\, dx$$

Let $u = \tan x$, $\Rightarrow du = \sec^2 x\, dx$

$$= \int u\, du - \int \tan x\, dx$$

$$= \frac{u^2}{2} - \log \cos x + c$$

$$= \frac{\tan^2 x}{2} - \log \cos x + c$$

Note:

① $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$

② $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$

③ $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

④ Evaluate $\int \sin 4x \cos 5x \, dx$

Soln: Given that $\int \sin 4x \cos 5x \, dx$.

W.K.T $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$

$$= \int \frac{1}{2}[\sin(4x-5x) + \sin(4x+5x)]\,dx$$

$$= \frac{1}{2}\int [\sin(-x) + \sin 9x]\,dx$$

$$= \frac{1}{2}\int [\sin x + \sin 9x]\,dx$$

$$I = \frac{1}{2}\left[\cos x - \frac{\cos 9x}{9}\right] + c$$

⑤ Evaluate $\int_0^{\pi/2} \cos^5 x \, dx$.

Soln: $\int_0^{\pi/2} \cos^5 x \, dx = \int_0^{\pi/2} \cos^4 x \, \cancel{dx} \cos x \, dx$

$$= \int_0^{\pi/2} (\cos^2 x)^2 \cos x \, dx$$

$$= \int_0^{\pi/2} (1-\sin^2 x)^2 \cos x \, dx$$

let $u = \sin x \Rightarrow du = \cos x \, dx$

when $x = 0 \Rightarrow u = \sin(0) \Rightarrow u = 0$

$x = \pi/2 \Rightarrow u = \sin \pi/2 \Rightarrow u = 1$

$$\int_0^{\pi/2} [1-\sin^2 x]^2 \cos x \, dx = \int_0^1 (1-u^2)^2 \, du$$

$$= \int_0^1 (1+u^4-2u^2) \, du$$

$$= \left[u + \frac{u^5}{5} - \frac{2u^3}{3} \right]_0^1$$

$$= \left[1 + 1/5 - 2/3 \right] - [0]$$

$$= \left[1 + 1/5 - 2/3 \right] = \left[\frac{15+3-10}{15} \right]$$

$$I = 8/15$$

(6) Evaluate $\int_0^{\pi} \sin^2 x \, dx$

Soln Given that $\int_0^{\pi} \sin^2 x \, dx$

$\left\{ \text{w.k.t } \sin^2\theta = \frac{1-\cos 2\theta}{2} \right\}$

$$= \int_0^{\pi} \left(\frac{1-\cos 2x}{2} \right) dx$$

$$= 1/2 \int_0^{\pi} (1-\cos 2x) \, dx$$

$$= 1/2 \left[x - \frac{\sin 2x}{2} \right]_0^{\pi}$$

$$= 1/2 \left\{ \left[\pi - \frac{\sin 2\pi}{2} \right] - [0 - \sin(0)] \right\}$$

$$= 1/2 [\pi - 0]$$

$$I = \pi/2$$

(7) Evaluate $\int_0^{\pi} \sin^2 x \cos^4 x \, dx$.

Soln: Given that $\int_0^{\pi} \sin^2 x \cos^4 x \, dx$

W.K.T $\sin^2\theta = \frac{1-\cos 2\theta}{2}$ & $\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$$\int_0^{\pi} \sin^2 x \cos^4 x \, dx = \int_0^{\pi} \sin^2 x (\cos^2 x)^2 dx$$

$$= \int_0^{\pi} \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{8}\int_0^{\pi} (1-\cos 2x)(1+\cos 2x)^2 dx$$

$$= \frac{1}{8}\int_0^{\pi} (1-\cos 2x)(1+\cos^2 2x + 2\cos 2x) dx$$

$$= \frac{1}{8}\int_0^{\pi} [1+\cos^2 2x + 2\cos 2x - \cos 2x - \cos^3 2x - 2\cos^2 2x] dx$$

$$= \frac{1}{8}\int_0^{\pi} [1+\cos 2x - \cos^2 2x - \cos^3 2x] dx$$

$$= \frac{1}{8}\int_0^{\pi} \left[1+\cos 2x - \left[\frac{1+\cos 4x}{2}\right] - \cos^2 2x \cos 2x\right] dx$$

$$= \frac{1}{8}\int_0^{\pi} \left[1+\cos 2x - \frac{1}{2} - \frac{\cos 4x}{2} - \left[\frac{1+\cos 4x}{2}\right]\cos 2x\right] dx$$

$$= \frac{1}{8}\int_0^{\pi} \left[1+\cos 2x - \frac{1}{2} - \frac{\cos 4x}{2} - \frac{\cos 2x}{2} - \frac{\cos 4x \cos 2x}{2}\right] dx$$

$$= \frac{1}{8}\int_0^{\pi}\left[1+\cos 2x-\frac{1}{2}-\frac{\cos 4x}{2}-\frac{\cos 2x}{2}-\frac{\cos 4x\cos 2x}{2}\right]dx$$

$\{$ W.k.T $\cos A\cos B = \frac{1}{2}[\cos(A+B)+\cos(A-B)]\}$

$$= \frac{1}{8}\int_0^{\pi}\left[1+\cos 2x-\frac{1}{2}-\frac{\cos 4x}{2}-\frac{\cos 2x}{2}-\left[\frac{\cos(4x+2x)+\cos(4x-2x)}{4}\right]\right]dx$$

$$= \frac{1}{8}\int_0^{\pi}\left[1+\cos 2x-\frac{1}{2}-\frac{\cos 4x}{2}-\frac{\cos 2x}{2}-\frac{\cos 6x}{4}-\frac{\cos 2x}{4}\right]dx$$

$$= \frac{1}{8}\int_0^{\pi}\left[1-\frac{1}{2}+\cos 2x-\frac{\cos 2x}{2}-\frac{\cos 2x}{4}-\frac{\cos 4x}{2}-\frac{\cos 6x}{4}\right]dx$$

$$= \frac{1}{8}\int_0^{\pi}\left[\frac{1}{2}+\frac{1}{4}\cos 2x-\frac{\cos 4x}{2}-\frac{\cos 6x}{4}\right]dx$$

$$= \frac{1}{8}\left[\frac{1}{2}x+\frac{1}{4}\frac{\sin 2x}{2}-\frac{\sin 4x}{8}-\frac{\sin 6x}{24}\right]_0^{\pi}$$

$$= \frac{1}{8}\left[\frac{1}{2}(\pi)+\frac{\sin 2\pi}{8}-\frac{\sin 4\pi}{8}-\frac{\sin 6\pi}{24}\right]-[0]$$

$$= \frac{1}{8}\left[\frac{1}{2}(\pi)+0\right]$$

$$I = \frac{\pi}{16}$$ //

Trigonometric substitution

Expression	Substitution	Identity
$\sqrt{a^2-x^2}$	$x = a\sin\theta\ ;\ -\pi/2 \le \theta \le \pi/2$	$\cos^2\theta = 1-\sin^2\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta\ ;\ -\pi/2 \le \theta \le \pi/2$	$1+\tan^2\theta = \sec^2\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta\ ;\ 0 \le \theta \le \pi/2$	$\sec^2\theta - 1 = \tan^2\theta$

① Evaluate $\int \frac{1}{\sqrt{a^2-x^2}}\,dx$

Soln:-

Given that $\int \frac{dx}{\sqrt{a^2-x^2}}$

Let us consider $x = a\sin\theta \implies dx = a\cos\theta\,d\theta$.

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{a\cos\theta\,d\theta}{\sqrt{a^2-a^2\sin^2\theta}}$$

$$= \int \frac{a\cos\theta\,d\theta}{\sqrt{a^2(1-\sin^2\theta)}} = \int \frac{a\cos\theta\,d\theta}{\sqrt{a^2\cos^2\theta}}$$

$$= \int \frac{\cancel{a}\,\cancel{\cos\theta}\,d\theta}{\cancel{a}\,\cancel{\cos\theta}} = \int d\theta$$

$$= [\theta]$$

W.K.T

$x = a\sin\theta$

$a\sin\theta = x$

$\sin\theta = x/a$

$\theta = \sin^{-1}(x/a)$

$$I = \sin^{-1}(x/a)$$

② Evaluate:- $\int \sqrt{a^2-x^2}\, dx$

Soln:-

Given that $\int \sqrt{a^2-x^2}\, dx$

$\sin^2\theta + \cos^2\theta = 1$

Let $x = a\sin\theta$

$dx = a\cos\theta\, d\theta$

$a\sin\theta = x$

$\sin\theta = x/a$

$\theta = \sin^{-1}(x/a)$

$\cos^2\theta = 1-\sin^2\theta$

$= 1-(x/a)^2$

$\cos^2\theta = (1-x^2/a^2)$

$\cos\theta = (1-x^2/a^2)^{1/2}$

$$\int \sqrt{a^2-x^2}\, dx = \int \sqrt{a^2-a^2\sin^2\theta}\; a\cos\theta\, d\theta$$

$$= \int \sqrt{a^2(1-\sin^2\theta)}\; a\cos\theta\, d\theta$$

$$= \int \sqrt{a^2\cos^2\theta}\; a\cos\theta\, d\theta$$

$$= \int a\cos\theta\; a\cos\theta\, d\theta = \int a^2\cos^2\theta\, d\theta$$

$$= a^2\int \cos^2\theta\, d\theta = a^2\int \left(\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= \frac{a^2}{2}\int (1+\cos 2\theta)\, d\theta$$

$$= \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]$$

W.K.T

$$\frac{\sin 2\theta}{2} = \sin\theta\cos\theta$$

$$= \frac{a^2}{2}\left[\theta + \sin\theta\cos\theta\right]$$

$$= \frac{a^2}{2}\left[\sin(x/a) + (x/a)(1-x^2/a^2)^{1/2}\right]$$

$$= a^2/2\left[\sin(x/a) + (x/a)\left(\frac{a^2-x^2}{a^2}\right)^{1/2}\right]$$

$$= a^2/2\left[\sin(x/a) + (x/a)\frac{\sqrt{a^2-x^2}}{a}\right]$$

$$= \frac{a^2}{2}\left[\sin(x/a) + (x/a)\frac{\sqrt{a^2-x^2}}{a}\right]$$

$$I = \frac{a^2}{2}\sin(x/a) + \frac{a^2}{2}\cdot\frac{x}{a}\frac{\sqrt{a^2-x^2}}{a}$$

$$I = \frac{a^2}{2}\sin(x/a) + \frac{x\sqrt{a^2-x^2}}{2}. //$$

(3) Evaluate: $\int \frac{x}{\sqrt{3-2x-x^2}}\, dx.$

Soln:

Given that $\int \frac{x}{\sqrt{3-2x-x^2}}\, dx$

Let us consider; $3-2x-x^2 = 3-(x^2+2x)$

$$= 3+1-(x^2+2x+1)$$

$$= 4-(x+1)^2$$

Now, put $u = x+1 \Rightarrow u-1 = x$

$du = dx$

$$\therefore \int \frac{x}{\sqrt{3-2x-x^2}}\, dx = \int \frac{u-1}{\sqrt{4-u^2}}\, du = \int \frac{(u-1)\, du}{\sqrt{4-u^2}}$$

Again, we put $u = 2\sin\theta$

$du = 2\cos\theta\, d\theta$

$$= \int \frac{(2\sin\theta - 1)}{\sqrt{4-(2\sin\theta)^2}}\, 2\cos\theta\, d\theta$$

$$= \int \frac{(2\sin\theta - 1)\, 2\cos\theta\, d\theta}{\sqrt{4-4\sin^2\theta}} = \int \frac{(2\sin\theta-1)\, 2\cos\theta\, d\theta}{\sqrt{4(1-\sin^2\theta)}}$$

$$= \int \frac{2\cos\theta\,(2\sin\theta - 1)\,d\theta}{\sqrt{4\cos^2\theta}}$$

$$= \int \frac{\cancel{2}\cos\theta\,(2\sin\theta - 1)}{\cancel{2}\cos\theta}\,d\theta = \int \frac{\cancel{\cos\theta}\,(2\sin\theta - 1)}{\cancel{\cos\theta}}\,d\theta$$

$$= \int (2\sin\theta - 1)\,d\theta = \left[-2\cos\theta - \theta\right] + C$$

$$= -\sqrt{4-u^2} - \sin^{-1}(u/2) + C$$

$$= -\sqrt{4-(x+1)^2} - \sin^{-1}\left(\frac{x+1}{2}\right) + C$$

$$= -\sqrt{4-(x^2+1+2x)} - \sin^{-1}\left(\frac{x+1}{2}\right) + C$$

$$I = -\sqrt{3-2x-x^2} - \sin^{-1}\left(\frac{x+1}{2}\right) + C$$

(4) Evaluate: $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$

Soln:

Given that $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$

Put $x = 2\tan\theta \Rightarrow dx = 2\sec^2\theta\,d\theta$

$$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx = \int \frac{2\sec^2\theta\,d\theta}{4\tan^2\theta\sqrt{4\tan^2\theta+4}}$$

$$= \int \frac{2\sec^2\theta\,d\theta}{4\tan^2\theta\sqrt{4(1+\tan^2\theta)}}$$

$$= \int \frac{2\sec^2\theta\, d\theta}{4\tan^2\theta\sqrt{4\sec^2\theta}} \qquad \{\text{W.K.T } 1+\tan^2\theta = \sec^2\theta\}$$

$$= \int \frac{\cancel{2}\sec^2\theta\, d\theta}{4\tan^2\theta(\cancel{2}\sec\theta)} = \int \frac{\sec^{\cancel{2}}\theta\, d\theta}{4\tan^2\theta\,\cancel{\sec\theta}}$$

$$= \frac{1}{4}\int \frac{\sec\theta\, d\theta}{\tan^2\theta} = \frac{1}{4}\int \frac{(1/\cos\theta)}{\left(\frac{\sin^2\theta}{\cos^2\theta}\right)} d\theta$$

$$= \frac{1}{4}\int \frac{1}{\cancel{\cos\theta}} \times \frac{\cos^{\cancel{2}}\theta}{\sin^2\theta} d\theta = \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta} d\theta$$

let $u = \sin\theta \Rightarrow du = \cos\theta\, d\theta$

$$= \frac{1}{4}\int \frac{du}{u^2} = \frac{1}{4}\int \frac{1}{u^2} du = \frac{1}{4}\left[-\frac{1}{u}\right] + C$$

$$= \frac{-1}{4u} = -\frac{1}{4}\left(\frac{1}{\sin\theta}\right) = -\frac{1}{4}\operatorname{Cosec}\theta + C$$

$$\underline{I} = -\frac{1}{4}\operatorname{Cosec}\theta + C$$

Put $x = 2\tan\theta \Rightarrow \tan\theta = x/2 \Rightarrow \theta = \tan^{-1}(x/2)$

Since $\tan\theta = x/2$; $\cot\theta = 2/x$

W.K.T $\operatorname{Cosec}^2\theta = 1 + \cot^2\theta$

$$\operatorname{Cosec}^2\theta = 1 + (2/x)^2 = 1 + \frac{4}{x^2} = \frac{x^2+4}{x^2}$$

$$\operatorname{Cosec}\theta = \sqrt{\frac{x^2+4}{x^2}} = \frac{\sqrt{x^2+4}}{x}$$

$$\therefore \int \frac{1}{x^2\sqrt{x^2+4}} = -\frac{1}{4}\operatorname{Cosec}\theta + C$$

$$= -\frac{1}{4}\frac{\sqrt{x^2+4}}{x} + C$$

(3) Evaluate: $\int \frac{1}{\sqrt{a^2-x^2}}\,dx$

Soln: Given that $\int \frac{1}{\sqrt{a^2-x^2}}\,dx$

Let $x = a\sin\theta$, $\Rightarrow dx = a\cos\theta\, d\theta$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{a\cos\theta\, d\theta}{\sqrt{a^2 - a^2\sin^2\theta}} = \int \frac{a\cos\theta\, d\theta}{\sqrt{a^2(1-\sin^2\theta)}}$$

$$= \int \frac{a\cos\theta\, d\theta}{\sqrt{a^2\cos^2\theta}} = \int \frac{\cancel{a\cos\theta}\, d\theta}{\cancel{a\cos\theta}}$$

$$= \int d\theta = \theta + C$$

$$I = \sin^{-1}(x/a) + C$$

$x = a\sin\theta$

$a\sin\theta = x$

$\sin\theta = x/a$

$\theta = \sin^{-1}(x/a)$.

Integrals of the form

TYPE-I

(i) $\int \frac{dx}{\sqrt{ax^2+bx+c}}$

① Evaluate: $\int \frac{dx}{\sqrt{2-3x+x^2}}$

Soln: Given that $\int \frac{dx}{\sqrt{2-3x+x^2}}$

Here, $x^2-3x+2 = (x-3/2)^2 - 1/4$

Put $u = x - 3/2$

$du = dx$

$\therefore \int \frac{dx}{\sqrt{2-3x+x^2}} = \int \frac{dx}{\sqrt{(x-3/2)^2 - 1/4}}$

$= \int \frac{du}{\sqrt{u^2-(1/2)^2}}$

$\left\{ \text{W.K.T} \int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}(x/a) \right\}$

$\int \frac{dx}{\sqrt{2-3x+x^2}} = \cosh^{-1}\left[\frac{u}{(1/2)}\right] + C$

$= \cosh^{-1}[2u] + C$

$= \cosh^{-1}[2(x-3/2)] + C$

$= \cosh^{-1}\left[\cancel{2}\left(\frac{2x-3}{\cancel{2}}\right)\right] + C$

$= \cosh^{-1}[2x-3] + C$ //

$(a-b)^2 = b^2$

$= a^2+2ab+b^2$

$(x-3/2)^2 - 1/4$

$x^2+9/4-2(3/2 x)-1/4$

$x^2-3x+9/4-1/4$

$x^2-3x+8/4$

x^2-3x+2.

② Evaluate $\int \frac{x}{\sqrt{x^2+x+1}}\,dx$

Soln:

Given that $\int \frac{x}{\sqrt{x^2+x+1}}\,dx$

Let us Consider $x = A\frac{d}{dx}[x^2+x+1] + B$

$$x = A(2x+1) + B$$

Equating the Coefficient of 'x' on both side.

$$1 = 2A \Rightarrow 2A = 1 \Rightarrow \boxed{A = 1/2}$$

Equating the constant terms on both side,

$$0 = A+B \Rightarrow B = -A \Rightarrow \boxed{B = -1/2}$$

$$\int \frac{x}{\sqrt{x^2+x+1}}\,dx = \int \frac{A(2x+1)+B}{\sqrt{x^2+x+1}}\,dx$$

$$= A\int \frac{(2x+1)}{\sqrt{x^2+x+1}}\,dx + B\int \frac{1}{\sqrt{x^2+x+1}}\,dx$$

$$= 1/2\int \frac{(2x+1)}{\sqrt{x^2+x+1}}\,dx - 1/2\int \frac{1}{\sqrt{x^2+x+1}}\,dx$$

$$= 1/2 \cdot 2\sqrt{x^2+x+1} - 1/2\int \frac{1}{(x^2+x+1)^{1/2}}\,dx$$

$$= \sqrt{x^2+x+1} - 1/2\int \frac{dx}{\left[(x+1/2)^2 + (3/4)\right]^{1/2}}$$

$$= \sqrt{x^2+x+1} - 1/2\int \frac{dx}{\left[(x+1/2)^2 + \left(\frac{\sqrt{3}}{2}\right)^2\right]^{1/2}}$$

W.K.T $\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}(x/a)$

$$= \sqrt{x^2+x+1} - \frac{1}{2} \sinh^{-1}\left[\frac{(x+1/2)}{(\sqrt{3}/2)}\right] + C$$

$$I = \sqrt{x^2+x+1} - \frac{1}{2} \sinh^{-1}\left[\frac{2x+1}{\sqrt{3}}\right] + C$$

③ Evaluate $\int \frac{3x-2}{\sqrt{4x^2-4x-5}} dx$.

Soln. Given that $\int \frac{3x-2}{\sqrt{4x^2-4x-5}}$

let us assume that, $3x-2 = A \frac{d}{dx}(4x^2-4x-5) + B$

$$3x-2 = A(8x-4) + B$$

Equating the Coefficient of 'x' on both side

$$3 = 8A \Rightarrow 8A = 3 \Rightarrow \boxed{A = 3/8}$$

Equating the Coefficient of Constant terms on both side

$$-2 = -4A + B$$

$$-2 = -\cancel{4}\left(\frac{3}{\cancel{8}_2}\right) + B \Rightarrow -2 = -3/2 + B$$

$$\Rightarrow -2 + 3/2 = B$$

$$\Rightarrow \frac{-4+3}{2} = B$$

$$-1/2 = B$$

$$\boxed{B = -1/2}$$

$$\int \frac{3x-2}{\sqrt{4x^2-4x-5}}\,dx = A\int \frac{d/dx(4x^2-4x-5)}{\sqrt{4x^2-4x-5}}\,dx + B\int \frac{dx}{\sqrt{4x^2-4x-5}}$$

$$= \frac{3}{8}\int \frac{d/dx(4x^2-4x-5)}{\sqrt{4x^2-4x-5}}\,dx - \frac{1}{2}\int \frac{dx}{\sqrt{4x^2-4x-5}}$$

$$= \frac{3}{\cancel{8}_4}\cdot\cancel{2}\sqrt{4x^2-4x-5} - \frac{1}{2}\int \frac{dx}{2\sqrt{x^2-x-5/4}}$$

$$= \frac{3}{4}\sqrt{4x^2-4x-5} - \frac{1}{2}\cdot\frac{1}{2}\int \frac{dx}{\sqrt{x^2-x-5/4}}$$

$$= \frac{3}{4}\sqrt{4x^2-4x-5} - \frac{1}{4}\int \frac{dx}{\sqrt{(x-\frac{1}{2})^2-(\frac{3}{2})}}$$

$$= \frac{3}{4}\sqrt{4x^2-4x-5} - \frac{1}{4}\int \frac{dx}{\sqrt{(x-\frac{1}{2})^2-(\sqrt{3/2})^2}}$$

$$= \frac{3}{4}\sqrt{4x^2-4x-5} - \frac{1}{4}\cosh^{-1}\left[\frac{(x-\frac{1}{2})}{\sqrt{3/2}}\right]$$

W.k.t $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}(x/a)$

$$= \frac{3}{4}\sqrt{4x^2-4x-5} - \frac{1}{4}\cosh^{-1}\left(\frac{2x-1}{2}\times\frac{\sqrt{2}}{\sqrt{3}}\right)$$

$$= \frac{3}{4}\sqrt{4x^2-4x-5} - \frac{1}{4}\cosh^{-1}\left(\frac{2x-1}{\sqrt{6}}\right) + C$$

TYPE-2 (i) $\int \sqrt{ax^2+bx+c}\; dx = \int \sqrt{x^2+a^2}\, dx$ and then Integrate

(ii) $\int (px+q)\sqrt{ax^2+bx+c}\, dx = A\int \sqrt{ax^2+bx+c}\cdot \frac{d}{dx}(ax^2+bx+c)$

$+ B\int \sqrt{ax^2+bx+c}$.

$= \frac{2}{3} A\,(ax^2+bx+c)^{3/2} + B\int \sqrt{ax^2+bx+c}\, dx$.

① Evaluate $\int \sqrt{x^2+2x+10}\, dx$

Soln:

Given that $\int \sqrt{x^2+2x+10}\, dx$

Let us Consider, $x^2+2x+10 = (x+1)^2+9$

$$\int \sqrt{x^2+2x+10}\, dx = \int \sqrt{(x+1)^2+9}\, dx$$

$$= \int \sqrt{(x+1)^2+3^2}\, dx$$

W.K.T $\int \sqrt{x^2+a^2}\, dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\sinh^{-1}(x/a)$

$$= \frac{1}{2}(x+1)\sqrt{x^2+2x+10} + \frac{9}{2}\sinh^{-1}\left(\frac{x+1}{3}\right)$$

$$= \left(\frac{x+1}{2}\right)\sqrt{x^2+2x+10} + \frac{9}{2}\sinh^{-1}\left(\frac{x+1}{3}\right) + c$$

② **Evaluate:** $\int (x+1)\sqrt{x^2-2x+2}\, dx$

Soln:

Given that $\int (x+1)\sqrt{x^2-2x+2}\, dx$

let us Consider, $x+1 = A \frac{d}{dx}(x^2-2x+2)+B$

$$x+1 = A(2x-2)+B$$

Equating the coefficient of 'x' on both sides,

$$1=2A \Rightarrow 2A=1 \Rightarrow \boxed{A=\tfrac{1}{2}}$$

Equating the constant terms, on both sides

$$1=-2A+B \Rightarrow 1=-\cancel{2}(\tfrac{1}{\cancel{2}})+B$$

$$\Rightarrow 1=-1+B$$

$$\Rightarrow 1+1=B \Rightarrow \boxed{B=2}$$

$$I = \tfrac{2}{3}A(x^2-2x+2)^{3/2}+B\int\sqrt{x^2-2x+2}\, dx$$

$$= \tfrac{\cancel{2}}{3}(\tfrac{1}{\cancel{2}})(x^2-2x+2)^{3/2}+2\int\sqrt{x^2-2x+2}\, dx$$

$$= \tfrac{1}{3}(x^2-2x+2)^{3/2}+2\int\sqrt{(x-1)^2+1}\, dx$$

W.K.T $\int\sqrt{x^2+a^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sinh^{-1}(x/a)$

$$= \tfrac{1}{3}(x^2-2x+2)^{3/2}+2\left[\left(\frac{x-1}{2}\right)\sqrt{(x-1)^2+1^2}+\frac{1}{2}\sinh^{-1}\frac{(x-1)}{1}\right]$$

$$= \tfrac{1}{3}(x^2-2x+2)^{3/2}+2\left[\frac{x-1}{2}\sqrt{x^2-2x+2}+\tfrac{1}{2}\sinh^{-1}(x-1)\right]$$

$$I = \tfrac{1}{3}(x^2-2x+2)^{3/2}+\left(\frac{x-1}{2}\right)\sqrt{x^2-2x+2}+\sinh^{-1}(x-1)+C$$ //

(3) Evaluate $\int x\sqrt{1-x^4}\,dx$

Soln

Given that $\int x\sqrt{1-x^4}\,dx$

Let us consider, $x^2 = u \Rightarrow 2x\,dx = du$

$$x\,dx = du/2$$

$$\int x\sqrt{1-x^4}\,dx = \int \sqrt{1-x^4}\;x\,dx = \int \sqrt{1-u^2}\,(\tfrac{1}{2}du)$$

$$= \tfrac{1}{2}\int \sqrt{1-u^2}\,du$$

W.K.T $\int \sqrt{a^2-x^2}\,dx = a^2/2\,\sin^{-1}(x/a) + x/2\,\sqrt{a^2-x^2} + C$

$$= \tfrac{1}{2}\left[\tfrac{1}{2}\sin^{-1}(u/1) + u/2\,\sqrt{1-u^2}\right] + C$$

$$= \tfrac{1}{2}\left[\tfrac{1}{2}\sin^{-1}(u) + u/2\,\sqrt{1-u^2}\right] + C$$

$$= \tfrac{1}{4}\left[\sin^{-1}(u) + u/2\,\sqrt{1-u^2}\right] + C$$

$$I = \tfrac{1}{4}\left[\sin^{-1}(x^2) + x^2/2\,\sqrt{1-x^4}\right] + C$$

TYPE-(5)

(i) $\int \frac{dx}{(ax^2+bx+c)} = \int \frac{dx}{(a^2+x^2)}$ and then Integrate.

(ii) $\int \frac{Px+Q}{(ax^2+bx+c)}\,dx = A\log(ax^2+bx+c) + B\int \frac{dx}{ax^2+bx+c}$.

① Evaluate:- $\int \frac{dx}{(x^2+2x+5)}$

Soln:-

Given $\int \frac{dx}{(x^2+2x+5)}$

$= \int \frac{dx}{[(x+1)^2+4]} = \int \frac{dx}{(x+1)^2+2^2}$

W.K.T $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}(x/a)$

$= \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)$

② Evaluate:- $\int \frac{dx}{4x^2-4x+2}$

Soln:

Given that $\int \frac{dx}{4x^2-4x+2}$ / let us consider, $4x^2-4x+2$

$= 4(x^2-4+\frac{1}{2})$

$= 4[(x-\frac{1}{2})^2+\frac{1}{4}]$

$= \int \frac{dx}{4[(x-\frac{1}{2})^2+\frac{1}{4}]}$

$= \frac{1}{4}\int \frac{dx}{(x-\frac{1}{2})^2+(\frac{1}{2})^2}$ $\left\{ \text{W.K.T} \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}x/a \right\}$

$= \frac{1}{4}\left(\frac{1}{\cancel{1/2}}\right)\tan^{-1}\left[\frac{x-\frac{1}{2}}{\frac{1}{2}}\right]$

$= \frac{1}{\cancel{4}}(\cancel{2})\tan^{-1}[2(x-\frac{1}{2})]$

$= \frac{1}{2}\tan^{-1}\left[\cancel{2}\frac{(2x-1)}{\cancel{2}}\right]$

$= \frac{1}{2}\tan^{-1}(2x-1)$

(3) Evaluate $\int \frac{2x+3}{x^2+x+1}\,dx$.

Sol/n: Given that $\int \frac{2x+3}{x^2+x+1}\,dx$

Let us consider $2x+3 = A\,\frac{d}{dx}(x^2+x+1) + B$

$$2x+3 = A(2x+1) + B$$

Equating the x terms on b.s
$\cancel{2} = \cancel{2}A \Rightarrow \boxed{A=1}$

Equating the constant terms on b.s
$3 = A+B$
$3 = 1+B \Rightarrow B = 3-1 \Rightarrow \boxed{B=2}$

{W.K.T $\int \frac{px+q}{ax^2+bx+c}\,dx = A\log(ax^2+bx+c) + B\int \frac{dx}{ax^2+bx+c}$

$$\therefore \int \frac{2x+3}{x^2+x+1}\,dx = (1)\log(x^2+x+1) + 2\int \frac{dx}{x^2+x+1}$$

$$= \log(x^2+x+1) + 2\int \frac{dx}{(x+\frac{1}{2})^2 + \frac{3}{4}}$$

$$= \log(x^2+x+1) + 2\int \frac{dx}{(x+\frac{1}{2})^2 + (\sqrt{3/4})^2}$$

$$= \log(x^2+x+1) + 2\int \frac{dx}{(x+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

{W.K.T $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}(x/a)$

$$= \log(x^2+x+1) + 2\left(\frac{1}{\frac{\sqrt{3}}{2}}\right)\tan^{-1}\left(\frac{(x+\frac{1}{2})}{\sqrt{3}/2}\right)$$

$$= \log(x^2+x+1) + 2\left(\frac{2}{\sqrt{3}}\right)\tan^{-1}\frac{2(x+\frac{1}{2})}{\sqrt{3}}$$

$$= \log(x^2+x+1) + \frac{4}{\sqrt{3}}\tan^{-1}\left[\frac{\cancel{2}\frac{(2x+1)}{\cancel{2}}}{\sqrt{3}}\right]$$

$$= \log(x^2+x+1) + \frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C.$$

III - Integration of Rational Functions By Partial Fractions

① Evaluate: $\int \frac{dx}{x^2-a^2}$

Soln: Given that $\int \frac{dx}{x^2-a^2}$

Let us consider, $\frac{1}{x^2-a^2} = \frac{1}{(x+a)(x-a)} = \frac{A}{x+a} + \frac{B}{x-a}$

$$= \frac{A(x-a)+B(x+a)}{(x+a)(x-a)}$$

Compare Nr on both side,

$$1 = A(x-a) + B(x+a) \longrightarrow ①$$

Put $\boxed{x=-a}$ in Eqn ①

$1 = A(-a-a)+B(0)$

$1 = A(-2a)$

$\boxed{A = -1/2a}$

Put $\boxed{x=a}$ in Eqn ①

$1 = A(0)+B(a+a)$

$1 = B(2a)$

$\boxed{B = 1/2a}$

$$\int \frac{dx}{x^2-a^2} = \int \frac{A}{x+a}dx + \int \frac{B\,dx}{(x-a)}$$

$$= \int \frac{-1/2a}{x+a}dx + \int \frac{1/2a\,dx}{x-a} = 1/2a \int \frac{-dx}{x+a} + \int \frac{dx}{x-a}$$

$$= 1/2a \int \frac{dx}{x-a} - \int \frac{dx}{x+a}$$

$$= 1/2a\left[\log(x-a) - \log(x+a)\right] + c \ 1/2a$$

$$= 1/2a \log\left(\frac{x-a}{x+a}\right)$$

(2) Evaluate: $\int \frac{dx}{x^2+3x+2}$

Sol^n: Given that $\int \frac{dx}{x^2+3x+2}$

lets consider, $\frac{1}{x^2+3x+2} = \frac{1}{(x+1)(x+2)}$

$$= \frac{A}{x+1} + \frac{B}{x+2} = \frac{A(x+2)+B(x+1)}{(x+1)(x+2)}$$

Compare Nr on b.s.

$$1 = A(x+2) + B(x+1) \longrightarrow (1)$$

Put $\boxed{x=-1}$ in Eqn (1)

$1 = A(-1+2) + B(0)$

$1 = A(1)$

$\boxed{A=1}$

Put $\boxed{x=-2}$ in Eqn (1)

$1 = A(0) + B(-2+1)$

$1 = B(-1)$

$\boxed{B=-1}$

$$\int \frac{dx}{x^2+3x+2} = \int \frac{dx}{(x+1).(x+2)}$$

$$= \int \left(\frac{A}{x+1} + \frac{B}{x+2}\right) dx = \int \left(\frac{1}{x+1} - \frac{1}{x+2}\right) dx$$

$$= \int \frac{dx}{x+1} - \int \frac{dx}{x+2}$$

$$= \log(x+1) - \log(x+2)$$

$$= \log\left[\frac{x+1}{x+2}\right].$$

(3) Evaluate: $\int \frac{\sec^2 x}{\tan^2 x + 3\tan x + 2}\,dx$

Soln:

Given that $\int \frac{\sec^2 x}{\tan^2 x + 3\tan x + 2}\,dx$

Put $u = \tan x \Rightarrow du = \sec^2 x\,dx$

$$\therefore \int \frac{\sec^2 x\,dx}{\tan^2 x + 3\tan x + 2} = \int \frac{du}{u^2+3u+2}$$

Let us consider, $\frac{1}{u^2+3u+2} = \frac{1}{(u+1)(u+2)} = \frac{A}{u+1} + \frac{B}{u+2}$

$$= \frac{A(u+2)+B(u+1)}{(u+1)(u+2)}$$

Compare N_r on both side,

$$1 = A(u+2) + B(u+1) \longrightarrow (1)$$

Put $\boxed{u=-1}$ in Eqn (1)

$1 = A(-1+2) + B(0)$

$\boxed{A=1}$

Put $\boxed{u=-2}$ in Eqn (1)

$1 = A(0) + B(-2+1)$

$1 = B(-1) \Rightarrow \boxed{B=-1}$

$$\int \frac{du}{u^2+3u+2} = \int \frac{du}{(u+1)(u+2)} = \int \left(\frac{A}{u+1} + \frac{B}{u+2}\right) du$$

$$= \int \left(\frac{1}{u+1} - \frac{1}{u+2}\right) du$$

$$= \log(u+1) - \log(u+2)$$

$$= \log\left[\frac{u+1}{u+2}\right]$$

$$= \log\left[\frac{\tan x+1}{\tan x+2}\right].$$

4) Evaluate $\int \frac{10\,dx}{(x-1)(x^2+9)}$.

Soln: Given that $\int \frac{10\,dx}{(x-1)(x^2+9)}$

Let us Consider, $\frac{10}{(x-1)(x^2+9)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+9} = \frac{A(x^2+9)+(Bx+C)(x-1)}{(x-1)(x^2+9)}$

Compare Nr on both side,

$$10 = A(x^2+9) + B(x+C)(x-1) \longrightarrow (1)$$

Put $\boxed{x=1}$ in Eqn (1)

$10 = A(1+9)$

$10 = 10A$

$\boxed{A=1}$

Compare coefficient of 'x^2'

$0 = A+B$

$B = -A$

$\boxed{B=-1}$

Compare coefficient of 'x'

$0 = -B+C$

$B = C$

$\boxed{C=-1}$

$$\therefore \frac{10}{(x-1)(x^2+9)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+9}$$

$$= \frac{1}{x-1} + \frac{-x-1}{x^2+9}$$

$$\frac{10}{(x-1)(x^2+9)} = \frac{1}{x-1} - \frac{(x+1)}{x^2+9}$$

$$\int \frac{10\,dx}{(x-1)(x^2+9)} = \int \frac{dx}{x-1} - \int \frac{x+1}{x^2+9}\,dx$$

$$= \int \frac{dx}{x-1} - \int \frac{x\,dx}{x^2+9} - \int \frac{dx}{x^2+9}$$

$$= \log(x-1) - \tfrac{1}{2}\log(x^2+9) - \tfrac{1}{3}\tan^{-1}(x/3).$$

5 Evaluate: $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx$

Soln:

Given that $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx$

$$
\begin{array}{r|l}
 & x+1 \\
\hline
x^3-x^2-x+1 & x^4+0x^3-2x^2+4x+1 \\
 & x^4-x^3-x^2+x \\
 & (-)\;(+)\;(+)\;(-) \\
\hline
 & x^3-x^2+3x+1 \\
 & x^3-x^2-x+1 \\
 & (-)\;(+)\;(+)\;(-) \\
\hline
 & 4x
\end{array}
$$

$\frac{x^4}{x^3} = x$

$\frac{x^3}{x^3} = 1$

$$\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = (x+1) + \frac{4x}{x^3-x^2-x+1}$$

$$= (x+1) + \frac{4x}{(x-1)^2(x+1)} \longrightarrow (A)$$

Now Consider, $\frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$

$$= \frac{A(x-1)(x+1) + B(x+1) + C(x-1)^2}{(x-1)^2(x+1)}$$

Compare Nr on both side,

$$4x = A(x-1)(x+1) + B(x+1) + C(x-1)^2 \longrightarrow (1)$$

Put $\boxed{x=1}$ in Eqn (1)

$4(1) = A(0) + B(1+1) + C(0)$

$4 = 2B$

$2B = 4$

$B = 4/2$

$\boxed{B=2}$

Put $\boxed{x=-1}$ in Eqn (1)

$4(-1) = A(0) + B(0) + C(-1-1)^2$

$-4 = C(-2)^2$

$-4 = 4C$

$\boxed{C=-1}$

Put $\boxed{x=0}$ in Eqn (1)

$0 = -A + B + C$

$A = B + C$

$= 2 - 1$

$\boxed{A=1}$

$$\frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

$$= \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1}$$

(A) $\Rightarrow \dfrac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = (x+1) + \dfrac{1}{x-1} + \dfrac{2}{(x-1)^2} - \dfrac{1}{x+1}$

∫ati, we get

$$\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx = \int (x+1)\,dx + \int \frac{dx}{x-1} + 2\int \frac{dx}{(x-1)^2} - \int \frac{dx}{x+1}$$

$$= \cancel{(x+1)^2}$$

$$= \frac{x^2}{2} + x + \log(x-1) + 2\,\frac{(x-1)^{-1}}{-1} - \log(x+1)$$

$$= \frac{x^2}{2} + x + \log(x-1) - 2(x-1)^{-1} - \log(x+1)$$

$$I = \frac{x^2}{2} + x + \log(x-1) - \frac{2}{(x-1)} - \log(x+1)$$

IMPROPER INTEGRALS

$$\int_a^\infty f(x)\,dx = \lim_{t \to \infty} \int_a^t f(x)\,dx$$

① Evaluate. $\int_1^\infty \frac{1}{x}\,dx$ and determine whether the integral is convergent or divergent.

Soln

Let $\int_1^\infty \frac{1}{x}\,dx = \lim_{t \to \infty} \int_1^t \frac{dx}{x}$

$$= \lim_{t \to \infty} \left[\log x\right]_1^t$$

$$= \lim_{t \to \infty} \left[\log t - \log(1)\right]$$

$$= \lim_{t \to \infty} \left[\log(t) - 0\right]$$

$$= \lim_{t \to \infty} \log(t)$$

$$= \infty \quad (\text{limit not Exist})$$

$\therefore \int_1^\infty \frac{1}{x}\,dx$ is divergent.

② Evaluate $\int_2^{\infty} 1/x^2 \, dx$ and determine whether the integral is convergent or divergent.

Soln.

$$\text{Let } \int_2^{\infty} 1/x^2 \, dx = \lim_{t\to\infty} \int_2^{t} 1/x^2 \, dx$$

$$= \lim_{t\to\infty} \left[-1/x\right]_2^t$$

$$= \lim_{t\to\infty} \left[-1/t + 1/2\right]$$

$$= -1/\infty + 1/2 = 0 + 1/2$$

$$= 1/2 \text{ (limit is exists)}$$

∴ The limit exists as a finite number and so the integral $\int_2^{\infty} (1/x^2)\, dx$ is convergent.

③ For what value of 'P' is $\int_1^{\infty} \frac{1}{x^P} dx$ convergent. Use the P- test.

Soln

$$\text{Let } \int_1^{\infty} \frac{1}{x^P} dx = \lim_{t\to\infty} \int_1^{t} \frac{1}{x^P} dx$$

$$= \lim_{t\to\infty} \left[\frac{x^{-P+1}}{-P+1}\right]_1^t$$

$$= \frac{1}{1-P} \lim_{t\to\infty} \left[\frac{1}{x^{P-1}}\right]_1^t$$

$$= \frac{1}{1-P} \lim_{t\to\infty} \left[\frac{1}{t^{P-1}} - 1\right] \longrightarrow (1)$$

Case (i) If $P>1 \Rightarrow P-1>0$

Here, $t \to \infty \Rightarrow t^{P-1} \to \infty$

$$\Rightarrow \frac{1}{t^{P-1}} \to 0$$

$$① \Rightarrow \int_1^\infty \frac{1}{x^P}\,dx = \frac{1}{1-P}[0-1]$$

$$= \frac{1}{1-P}(-1) = \frac{1}{P-1} < \infty$$

$\therefore \int_1^\infty \frac{1}{x^P}\,dx$ is Convergent if $P>1$.

Case (ii) If $P<1 \Rightarrow P-1<0 \Rightarrow 1-P>0$

Here, $t \to \infty \Rightarrow t^{1-P} \to \infty$

$$\Rightarrow t^{P-1} \to 0 \Rightarrow \frac{1}{t^{P-1}} \to \infty$$

$$① \Rightarrow \int_1^\infty \frac{1}{x^P}\,dx = \frac{1}{1-P}(\infty)$$

$$= \infty$$

$\therefore \int_1^\infty \frac{1}{x^P}\,dx$ is divergent if $P<1$.

Case (iii) If $P=1$, $\int_1^\infty \frac{1}{x^P}\,dx = \int_1^\infty \frac{1}{x}\,dx$

$$= [\log x]_1^\infty$$

$$= \log(\infty) - \log(1)$$

$$= \infty$$

$\therefore \int_1^\infty \frac{1}{x^P}\,dx$ is divergent if $P=1$.

(1) Determine whether each integral is convergent (or) divergent Evaluate those that are convergent.

(a) $\int_{-\infty}^{0} e^{x}\,dx$

Soln:

Let $\int_{-\infty}^{0} e^{x}\,dx = \lim_{t\to-\infty}\int_{t}^{0} e^{x}\,dx = \lim_{t\to-\infty}\left[e^{x}\right]_{t}^{0}$

$= \lim_{t\to-\infty}(e^{0}-e^{t}) = \lim_{t\to-\infty}(1-e^{t}]$

$= 1-e^{-\infty} = 1-0 = 1$

$\therefore \int_{-\infty}^{0} e^{x}\,dx$ is convergent.

(b) $\int_{0}^{\infty} e^{x}\,dx$

Soln

Let $\int_{0}^{\infty} e^{x}\,dx = \lim_{t\to\infty}\int_{0}^{t} e^{x}\,dx = \lim_{t\to\infty}\left[e^{x}\right]_{0}^{t}$

$= \lim_{t\to\infty}\left[e^{t}-e^{0}\right] = \lim_{t\to\infty}\left[e^{t}-1\right]$

$= e^{\infty}-1 = \infty$

$\therefore \int_{0}^{\infty} e^{x}\,dx$ is divergent.

(5) Evaluate: $\int_{-1}^{1} \frac{dx}{x}$.

Sol/n: Given that $\int_{-1}^{1} \frac{dx}{x}$

we first note that the given integral is improper because $f(x) = \frac{1}{x}$ has the vertical asymptote $x=0$, we have

$$\int_{-1}^{1} \frac{dx}{x} = \int_{-1}^{0} \frac{dx}{x} + \int_{0}^{1} \frac{dx}{x}$$

Since, $\int_{0}^{1} \frac{dx}{x} = \lim_{t\to 0^+} \int_{t}^{1} \frac{dx}{x} = \lim_{t\to 0^+} \left[\log x\right]_t^1$

$$= \lim_{t\to 0^+} \left[\log(1) - \log(t)\right] = \lim_{t\to 0} \left[0 - \log(t)\right]$$

$$= \lim_{t\to 0} \log t = \infty$$

it follows that $\int_{-1}^{1} \frac{dx}{x}$ is divergent.

(6) Evaluate $\int_{1}^{2} \frac{dx}{1-x} dx$

Sol/n: Given that $\int_{1}^{2} \frac{dx}{1-x}$

we first note that the given integral is improper because $f(x) = \frac{1}{1-x}$ has the vertical asymptote $x=1$.

$$\int_{1}^{2} \frac{dx}{1-x} = \lim_{t\to 1^+} \int_{t}^{2} \frac{dx}{1-x} = \lim_{t\to 1^+} \left[\log(1-x)\right]_t^2$$

$$= \lim_{t\to 1^+} \left[\log(1-2) - \log(1-t)\right]$$

$$= \lim_{t\to 1^+} \left[\log(-1) - \log(1-t)\right] = -\infty$$

it follows that $\int_{1}^{2} \frac{dx}{1-x}$ is divergent.

MA3151-MATRICES AND CALCULUS

UNIT-5

MULTIPLE INTEGRALS

Mr.V.PRAKASH,M.Sc;M.Phil;B.Ed;
Assistant Professor
Department of Mathematics

UNIT - 5

MULTIPLE INTEGRALS

Chapter - 5.1 [Double integration in cartesian co-ordinates]

Example - ① Evaluate $\int_0^1 \int_1^2 x(x+y)\,dy\,dx$

Soln:

Given that $\int_0^1 \int_1^2 x(x+y)\,dy\,dx$

$$= \int_0^1 \int_1^2 (x^2+xy)\,dy\,dx$$

$$= \int_0^1 \left[x^2y + \frac{xy^2}{2}\right]_1^2 dx$$

$$= \int_0^1 \left\{\left[x^2(2) + x\left(\frac{2^2}{2}\right)\right] - \left[x^2(1) + x\left(\frac{1}{2}\right)\right]\right\} dx$$

$$= \int_0^1 \left[2x^2 + \frac{4x}{2} - x^2 - x/2\right] dx$$

$$= \int_0^1 \left(x^2 + 3x/2\right) dx$$

$$= \left[\frac{x^3}{3} + \frac{3x^2}{4}\right]_0^1$$

$$= \left[1/3 + 3/4\right] - [0]$$

$$= \frac{4+9}{12}$$

$$\boxed{I = \frac{13}{12}}$$

Example - (2) Evaluate $\int_2^3\int_1^2 \frac{1}{xy}\,dx\,dy$

Sol^n:

$$\int_2^3\int_1^2 \frac{1}{xy}\,dx\,dy = \int_2^3\int_1^2 \frac{1}{x}\,\frac{1}{y}\,dx\,dy$$

$$= \int_2^3 1/y\left[\log x\right]_1^2 dy = \int_2^3 1/y\left[\log(2)-\log(1)\right]dy$$

$$= \int_2^3 1/y\left[\log(2)-0\right]dy = \int_2^3 1/y\,\log(2)\,dy$$

$$= \log(2)\int_2^3 1/y\,dy = \log(2)\left[\log(y)\right]_2^3$$

$$= \log(2)\left[\log(3)-\log(2)\right]$$

$$I = \log(2)\,\log(3/2)$$

Example -(3) $\int_0^1\int_0^{x^2} e^{y/x}\,dy\,dx$

Sol^n.

$$\int_0^1\int_0^{x^2} e^{y/x}\,dy\,dx = \int_0^1\left[\frac{e^{y/x}}{1/x}\right]_0^{x^2} dx = \int_0^1\left[\frac{e^{x^2/x}}{1/x}\right]-\left[\frac{e^0}{1/x}\right]dx$$

$$= \int_0^1\left[\frac{e^{\cancel{x}}}{1/x} - 1/{1/x}\right]dx = \int_0^1 x\left[e^x-1\right]dx$$

$$= \int_0^1\left[xe^x-x\right]dx = \int_0^1 xe^x\,dx - \int_0^1 x\,dx$$

$$= \left[xe^x-(1)e^x\right]_0^1 - \left[x^2/2\right]_0^1$$

$$= \left[(1)\cancel{e^1}-\cancel{e^1}\right]-\left[0-e^0\right]-1/2$$

$$= e^0 - 1/2 = 1-1/2 = 1/2 \qquad \boxed{I = 1/2}$$

Example - (4) Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} y \, dy \, dx$

Sol^n:

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} y \, dy \, dx = \int_0^a \left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \frac{\left(\sqrt{a^2-x^2}\right)^2}{2} dx = \int_0^a \left(\frac{a^2-x^2}{2}\right) dx$$

$$= \frac{1}{2} \int_0^a (a^2-x^2) dx = \frac{1}{2}\left[a^2 x - \frac{x^3}{3}\right]_0^a$$

$$= \frac{1}{2}\left\{\left[a^2(a) - a^3/3\right] - [0]\right\}$$

$$= \frac{1}{2}\left[a^3 - a^3/3\right] = \frac{1}{2}\left[\frac{3a^3 - a^3}{3}\right]$$

$$I = \frac{1}{\cancel{2}}\left[\frac{\cancel{2}a^3}{3}\right] \Rightarrow \boxed{I = \frac{a^3}{3}}$$

Example - (5) Find the value of $\int_0^\infty \int_0^y \frac{e^{-y}}{y} \, dx \, dy$

Sol^n:

$$\int_0^\infty \int_0^y \frac{e^{-y}}{y} \, dx \, dy = \int_0^\infty \frac{e^{-y}}{y} [x]_0^y \, dy = \int_0^\infty \frac{e^{-y}}{\cancel{y}} \cancel{(y)} \, dy$$

$$= \int_0^\infty e^{-y} \, dy = \left[\frac{e^{-y}}{-1}\right]_0^\infty$$

$$= -\left[e^{-y}\right]_0^\infty = -\left[e^{-\infty} - e^{0}\right]$$

$$= -[0-1] = -(-1)$$

$$\boxed{I = 1}$$

Example - (6) Evaluate $\int_{1}^{2}\int_{0}^{x} \frac{1}{x^2+y^2}\,dx\,dy$

Soln:

$$\int_{1}^{2}\int_{0}^{x} \frac{1}{x^2+y^2}\,dx\,dy$$

$$\left\{ \text{W.K.T } \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}(x/a) \right\}$$

$$= \int_{1}^{2}\int_{0}^{x} \frac{1}{x^2+y^2}\,dy\,dx$$

$$= \int_{1}^{2} \left[\frac{1}{x}\tan^{-1}(y/x)\right]_{0}^{x} dx = \int_{1}^{2} \left[\frac{1}{x}\tan^{-1}(x/x)\right] - \left[\frac{1}{x}\tan^{-1}(0)\right] dx$$

$$= \int_{1}^{2} \left[\frac{1}{x}\tan^{-1}(1) - \frac{1}{x}\tan(0)\right] dx = \int_{1}^{2} \frac{1}{x}\left[\tan^{-1}1 - \tan^{-1}0\right] dx$$

$$= \int_{1}^{2} \frac{1}{x}\left[\pi/4 - 0\right] dx = \int_{1}^{2} \frac{1}{x}(\pi/4)\,dx = \pi/4 \int_{1}^{2} \frac{1}{x}\,dx$$

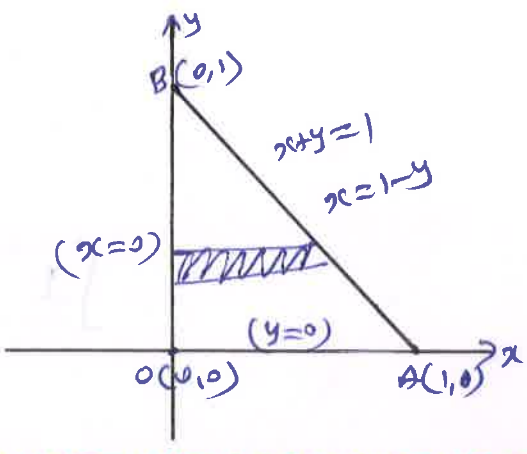
$$= \pi/4\left[\log(x)\right]_{1}^{2} = \pi/4\left[\log(2) - \log(1)\right] = \pi/4\left[\log(2) - 0\right]$$

$$\boxed{I = \pi/4 \log(2)}$$

Example - (7) Evaluate $\iint xy\,dx\,dy$ over the region in the positive quadrant for which $x+y \le 1$.

Soln:

The region of integration is the area bounded by the lines $x=0$; $y=0$; $x+y=1$



The limits are,

$y=0$ to $y=1$

$x=0$ to $x=1-y$

$$\iint xy\,dx\,dy = \int_0^1\int_0^{1-y} xy\,dx\,dy$$

$$= \int_0^1 \left[\frac{yx^2}{2}\right]_0^{1-y} dy = \int_0^1 \left[\frac{y(1-y)^2}{2} - (0)\right] dy$$

$$= \frac{1}{2}\int_0^1 y(1-y)^2\,dy = \frac{1}{2}\int_0^1 y[1+y^2-2y]\,dy$$

$$= \frac{1}{2}\int_0^1 (y+y^3-2y^2)\,dy = \frac{1}{2}\left[\frac{y^2}{2}+\frac{y^4}{4}-\frac{2y^3}{3}\right]_0^1$$

$$= \frac{1}{2}\left[\frac{1}{2}+\frac{1}{4}-\frac{2}{3}\right] = \frac{1}{2}\left[\frac{6+3-8}{12}\right]$$

$$= \frac{1}{2}\left[\frac{1}{12}\right]$$

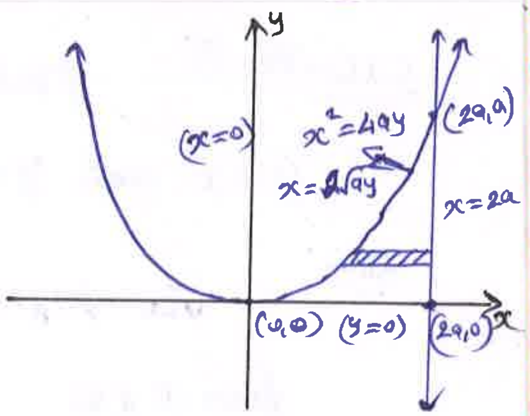
$$\boxed{I = \frac{1}{24}}$$

Example-8 Evaluate $\iint xy\,dx\,dy$, where R is the region bounded by x-axis $x=2a$, and $x^2=4ay$.

Soln.

The region bounded by $x=2a$ and $x^2=4ay \rightarrow (2)$

$\hookrightarrow (1)$



use (1) in (2)

(2) $\Rightarrow x^2=4ay$

$(2a)^2=4ay \Rightarrow 4a^2=4ay$

$a=y \Rightarrow \boxed{y=a}$

Put $y=a$ in Eqn (2)

(2) $\Rightarrow x^2=4a(a)$

$x^2=4a^2$

$\boxed{x=2a}$

Limits:

$y=0$ to $y=a$

$x=2\sqrt{ay}$ to $x=2a$

$x^2=4ay$

$x=\sqrt{4ay}$

$x=2\sqrt{ay}$

$$\iint xy\,dx\,dy = \int_0^a \int_{2\sqrt{ay}}^{2a} xy\,dx\,dy$$

$$= \int_0^a \left[\frac{yx^2}{2}\right]_{2\sqrt{ay}}^{2a} dy = \frac{1}{2}\int_0^a y\,[x^2]_{2\sqrt{ay}}^{2a}\,dy$$

$$= \frac{1}{2}\int_0^a y\left[(2a)^2 - (2\sqrt{ay})^2\right] dy$$

$$= \frac{1}{2}\int_0^a y\,[4a^2 - 4ay]\,dy$$

$$= \frac{1}{2}\int_0^a [4a^2y - 4ay^2]\,dy$$

$$= \frac{1}{2}\left[\frac{4a^2y^2}{2} - \frac{4ay^3}{3}\right]_0^a = \frac{1}{2}\left[\frac{4a^2(a^2)}{2} - \frac{4a(a^3)}{3}\right]$$

$$= \frac{1}{2}\left[\frac{4a^4}{2} - \frac{4a^4}{3}\right] = \frac{\cancel{4}^{2}a^4}{\cancel{2}}\left[\frac{1}{2} - \frac{1}{3}\right]$$

$$= 2a^4\left[\frac{3-2}{6}\right] = \cancel{2}a^4\left(\frac{1}{\cancel{6}_3}\right) = a^4\left(\frac{1}{3}\right)$$

$$\boxed{I = a^4/3}$$

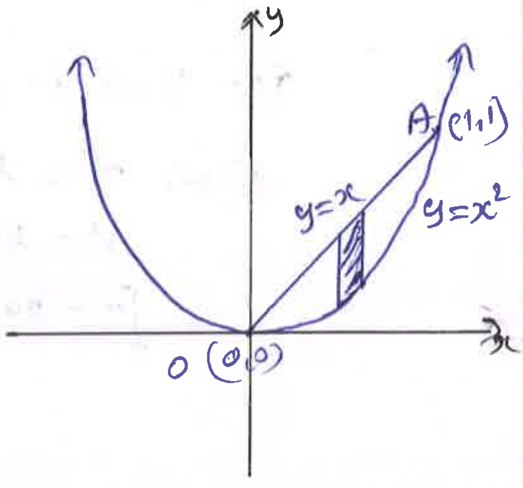
Example-(9) Evaluate $\iint xy(x+y)\,dx\,dy$ over the area between $y = x^2$ and $y = x$.

Soln- The region bounded by $y = x^2 \longrightarrow (1)$ and $y = x \longrightarrow (2)$

from (1) & (2)

$$x = x^2 \Rightarrow x^2 - x = 0$$
$$x(x-1) = 0$$
$$\boxed{x=0};\ x-1=0$$
$$\boxed{x=1}$$

$$\therefore\ y = x \Rightarrow \boxed{y=1}$$



Limit:

$x=0$ to $x=1$

$y=x^2$ to $y=x$

$$\int_0^1\int xy(x+y)\,dx\,dy = \iint (x^2y+xy^2)\,dy\,dx$$

$$= \int_0^1\int_{x^2}^{x} (x^2y+xy^2)\,dy\,dx$$

$$= \int_0^1 \left[\frac{x^2y^2}{2}+\frac{xy^3}{3}\right]_{x^2}^{x} dx$$

$$= \int_0^1 \left\{\left[\frac{x^2(x^2)}{2}+\frac{x(x^3)}{3}\right]-\left[\frac{x^2(x^2)^2}{2}+\frac{x(x^2)^3}{3}\right]\right\} dx$$

$$= \int_0^1 \left\{\left(\frac{x^4}{2}+\frac{x^4}{3}\right)-\left(\frac{x^2(x^4)}{2}+\frac{x(x^6)}{3}\right)\right\} dx$$

$$= \int_0^1 \left(\frac{x^4}{2}+\frac{x^4}{3}-\frac{x^6}{2}+\frac{x^7}{3}\right) dx$$

$$= \left[\frac{x^5}{2\times4}+\frac{x^5}{3\times5}-\frac{x^7}{2\times7}+\frac{x^8}{3\times8}\right]_0^1$$

$$= \left[\frac{x^5}{8}+\frac{x^5}{15}-\frac{x^7}{14}+\frac{x^8}{24}\right]_0^1$$

$$= \left[\frac{1}{8}+\frac{1}{15}-\frac{1}{14}+\frac{1}{24}\right]$$

$$\boxed{I = 3/56}$$

Example 10 Evaluate $\iint (x^2+y^2)\,dx\,dy$ over the region for which $x, y \geq 0$ and $x+y \leq 1$.

Soln:

The given region are $x, y \geq 0$, and $x+y \leq 1$

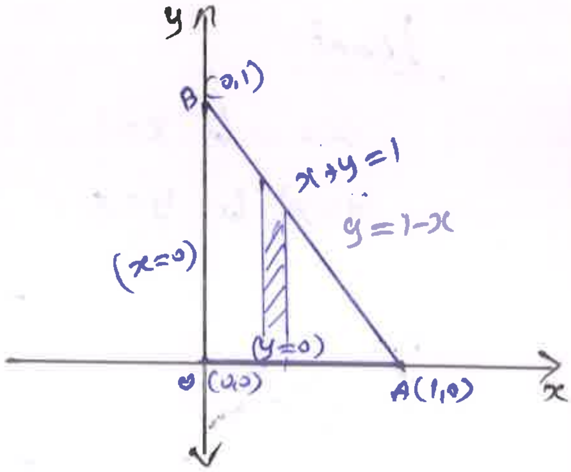
ie, $x=0$; $y=0$; $x+y=1$

(8)

Limits:-

$x = 0$ to $x = 1$

$y = 0$ to $y = 1-x$



$$\iint (x^2+y^2)\,dx\,dy = \int_0^1\int_0^{1-x} (x^2+y^2)\,dy\,dx$$

$$= \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_0^{1-x} dx$$

$$= \int_0^1 \left[x^2(1-x) + \frac{(1-x)^3}{3}\right] dx$$

$$= \int_0^1 \left[x^2 - x^3 + \frac{(1-x)^3}{3}\right] dx$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4} + \frac{(1-x)^4}{3\times(-4)}\right]_0^1$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4} + \frac{(1-x)^4}{-12}\right]_0^1$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4} - \frac{(1-x)^4}{12}\right]_0^1$$

$$= \left[\frac{1}{3} - \frac{1}{4} - \frac{(1-1)^4}{12}\right] - \left[0 - \frac{(1-0)^4}{12}\right]$$

$$= \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{12}\right)$$

$$= \frac{1}{3} - \frac{1}{4} + \frac{1}{12} = \frac{4-3+1}{12} = \frac{5-3}{12} = \frac{2}{12}$$

$$\boxed{I = \frac{1}{6}}$$

Example-(11) Find the value of $\int_0^\infty \int_0^y \frac{e^{-y}}{y}\,dx\,dy$

Soln:

Given $\int_0^\infty \int_0^y \frac{e^{-y}}{y}\,dx\,dy$

$$= \int_0^\infty \frac{e^{-y}}{y}[x]_0^y\,dy = \int_0^\infty \frac{e^{-y}}{\cancel{y}}(\cancel{y})\,dy$$

$$= \int_0^\infty e^{-y}\,dy = \left[\frac{e^{-y}}{-1}\right]_0^\infty = \left[-e^{-y}\right]_0^\infty$$

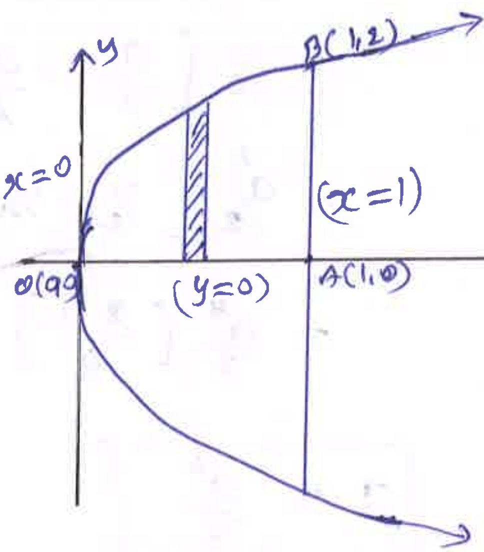
$$= \left[(-e^{-\infty}) - (-e^{0})\right] = [0-(-1)] = 1$$

$$\boxed{I = 1}$$

Example-(12) Find the limits of integration in the double integral $\iint_R f(x,y)\,dx\,dy$, where R is in the first quadrant and bounded by $x=1$, $y=0$, $y^2=4x$.

Soln

Given region $x=1$; $y=0$; $y^2=4x$



$y^2=4x$
$y=\sqrt{4x}$
$y=2\sqrt{x}$

ie, $y^2=4x$

$y^2=4(1)$

$y^2=4 \Rightarrow y=\pm 2$

$\boxed{y=2}$

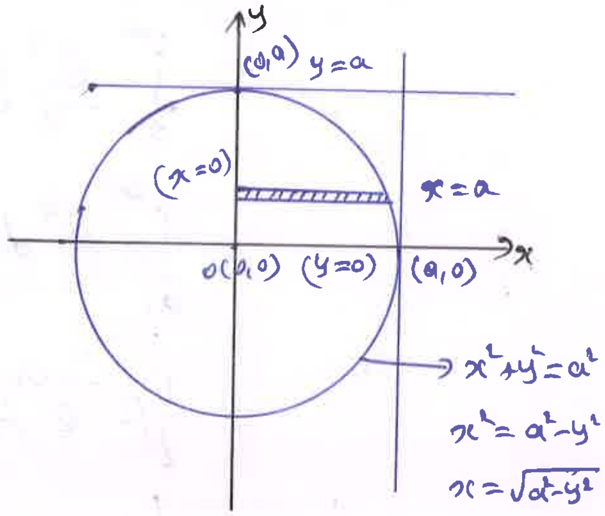
Limits:

$x=0$ to $x=1$

$y=0$ to $y=2\sqrt{x}$.

Example (13) Evaluate $\iint xy\,dx\,dy$ over the positive quadrant of the circle $x^2+y^2=a^2$.

Soln: Given that positive quadrant of the circle $x^2+y^2=a^2$



Limits:

$y=0$ to $y=a$

$x=0$ to $x=\sqrt{a^2-y^2}$

$$\iint xy\,dx\,dy = \int_0^a \int_0^{\sqrt{a^2-y^2}} xy\,dx\,dy$$

$$= \int_0^a y\left[\frac{x^2}{2}\right]_0^{\sqrt{a^2-y^2}} dy$$

$$= \int_0^a y\left[\frac{\sqrt{a^2-y^2}}{2}\right]^2 dy$$

$$= \frac{1}{2}\int_0^a y(a^2-y^2)\,dy = \frac{1}{2}\int_0^a (a^2y-y^3)\,dy$$

$$= \frac{1}{2}\left[\frac{a^2y^2}{2}-\frac{y^4}{4}\right]_0^a = \frac{1}{2}\left[\frac{a^2(a^2)}{2}-\frac{a^4}{4}\right]$$

$$= \frac{1}{2}\left[\frac{a^4}{2}-\frac{a^4}{4}\right]$$

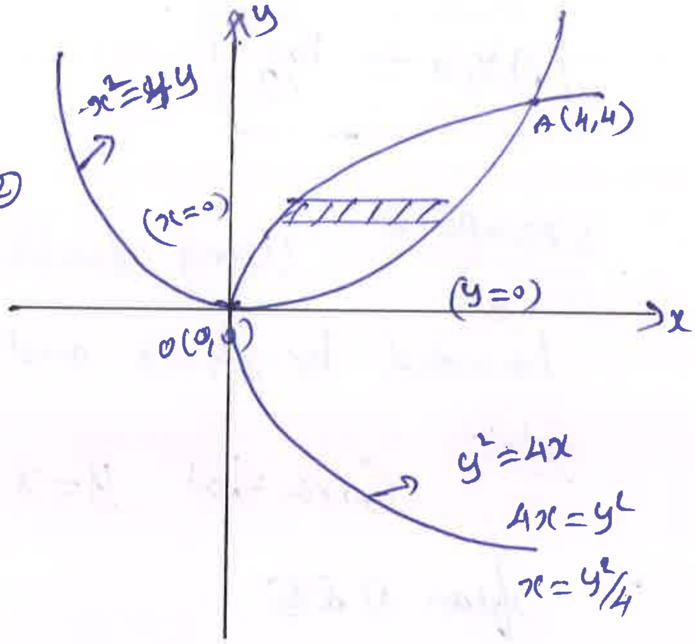
$$= \frac{a^4}{2}\left[\frac{1}{2}-\frac{1}{4}\right]$$

$$= \frac{a^4}{2}\left[\frac{2-1}{4}\right] = \frac{a^4}{2}\left[\frac{1}{4}\right]$$

$$\boxed{I = \frac{a^4}{8}}$$

Find the Area using double integration

Example-1 Find by double integration, the area between the parabolas $y^2 = 4x$ and $x^2 = 4y$.



Sol^n:

Given that $y^2 = 4x$ and $x^2 = 4y$
$\hookrightarrow$ (1) $\hookrightarrow$ (2)

from (1) & (2)

$$x^2 = 4y$$
$$x = \sqrt{4y}$$
$$x = 2\sqrt{y}$$

$$y^2 = 4x$$
$$y^2 = 4(2\sqrt{y})$$
$$y^2 = 8\sqrt{y}$$
$$(y^2)^2 = (8\sqrt{y})^2$$
$$y^4 = 64y \Rightarrow y^4 - 64y = 0$$
$$y(y^3 - 64) = 0$$
$$\boxed{y=0}; \quad y^3 - 64 = 0$$
$$y^3 = 64$$
$$\boxed{y=4}$$

Limits:

$y = 0$ to $y = 4$

$x = y^2/4$ to $x = 2\sqrt{y}$

$$\text{Area} = \iint dx\,dy$$
$$= \int_0^4 \int_{y^2/4}^{2\sqrt{y}} dx\,dy = \int_0^4 [x]_{y^2/4}^{2\sqrt{y}} dy$$
$$= \int_0^4 \left[2\sqrt{y} - y^2/4\right] dy = \int_0^4 \left[2y^{1/2} - \frac{y^2}{4}\right] dy$$
$$= \left[\frac{2y^{3/2}}{3/2} - \frac{y^3}{12}\right]_0^4 = \left[\frac{4y^{3/2}}{3} - \frac{y^3}{12}\right]_0^4$$

$$= \left[\frac{4y^{3/2}}{3} - \frac{y^3}{12}\right]_0^4 = \left[\frac{4(4)^{3/2}}{3} - \frac{(4)^3}{12}\right]$$

$$= \frac{4(8)}{3} - \frac{64}{12} = \frac{32}{3} - \frac{64}{12} = \frac{32}{3} - \frac{16}{3}$$

$$\boxed{\text{Area} = 16/3}$$

Example-② Using double integral, find the area bounded by $y=x$ and $y=x^2$.

Soln: Given that $y=x \longrightarrow ①$ and $y=x^2 \longrightarrow ②$

from ① & ②

② $\Rightarrow$ $y=x^2$

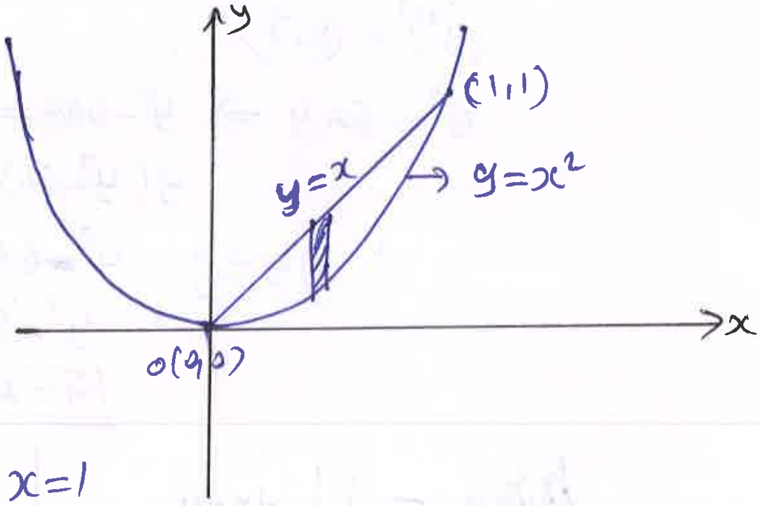
$x=x^2$

$x^2-x=0$

$x(x-1)=0$

$\boxed{x=0}$; $x-1=0$

$\boxed{x=1}$



Limits:

$x=0$ to $x=1$

$y=x$ to $y=x^2$

$$\text{Required Area} = \iint dy\,dx$$

$$= \int_0^1 \int_{x^2}^{x} dy\,dx = \int_0^1 [y]_{x^2}^{x} dx$$

$$= \int_0^1 (x-x^2)\,dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

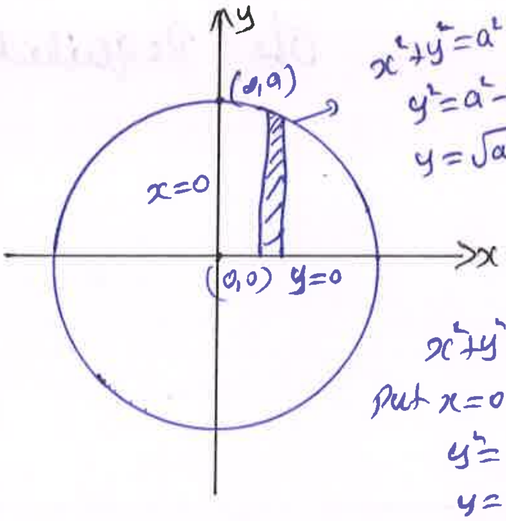
$$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= \left[\tfrac{1}{2} - \tfrac{1}{3}\right] - [0]$$

$$= \tfrac{1}{2} - \tfrac{1}{3} = \frac{3-2}{6}$$

$$\boxed{\text{Area} = 1/6}$$

Example-③ Find by double integration the area of the circle $x^2+y^2=a^2$.



$x^2+y^2=a^2$
$y^2=a^2-x^2$
$y=\sqrt{a^2-x^2}$

$x^2+y^2=a^2$
Put $x=0$
$y^2=a^2$
$y=\pm a$
$\boxed{y=a}$

Solve: Given that $x^2+y^2=a^2$

Limits:
$x=0$ to $x=a$
$y=0$ to $y=\sqrt{a^2-x^2}$

Area of the 1^{st} quadrant

$$I = \iint dy\,dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} dy\,dx = \int_0^a [y]_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \sqrt{a^2-x^2}\,dx \qquad \left\{ \text{W.K.T} \int \sqrt{a^2-x^2}\,dx = \tfrac{x}{2}\sqrt{a^2-x^2} + \tfrac{a^2}{2}\sin^{-1}(x/a) \right\}$$

$$= \left[\tfrac{x}{2}\sqrt{a^2-x^2} + \tfrac{a^2}{2}\sin^{-1}(x/a)\right]_0^a$$

$$= \left[\tfrac{a}{2}\sqrt{a^2-a^2} + \tfrac{a^2}{2}\sin^{-1}(a/a)\right] - \left[\tfrac{0}{2}\sqrt{a^2-0} + \tfrac{a^2}{2}\sin^{-1}(0)\right]$$

$$= \left[\frac{a}{2}(0) + \frac{a^2}{2}\sin^{-1}(1)\right] - [0+0]$$

$$= 0 + \frac{a^2}{2}\sin^{-1}(1)$$

$$= \frac{a^2}{2}\sin^{-1}(1)$$

$$= \frac{a^2}{2}\left(\frac{\pi}{2}\right)$$

$$\boxed{I = \frac{\pi a^2}{2}}$$

$\therefore$ The Required Area of the Circle is

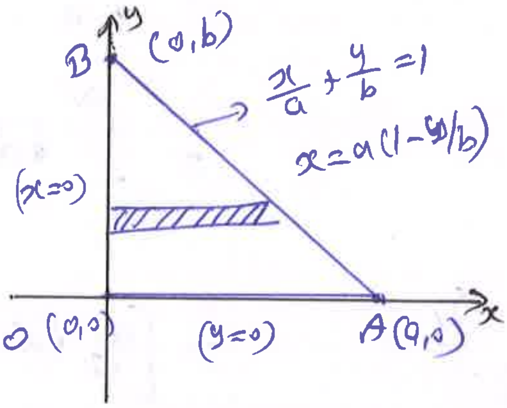
$$= 4 \times I$$

$$= {}^{2}\cancel{4} \times \frac{\pi a^2}{\cancel{2}}$$

$$\boxed{= 2\pi a^2}$$

Example - (4) Evaluate $\iint xy\,dx\,dy$ over the region in the positive quadrant bounded by $\frac{x}{a} + \frac{y}{b} = 1$.

Sol/n: Given that $\frac{x}{a} + \frac{y}{b} = 1 \longrightarrow (1)$



Put $\boxed{x=0}$ in Eq (1)

$$\frac{0}{a} + y/b = 1$$

$$y/b = 1$$

$$\boxed{y=b}$$

and $\frac{x}{a} + \frac{y}{b} = 1$

$\frac{x}{a} = 1 - y/b$

$\boxed{x = a(1 - y/b)}$

<u>Limits:</u>

$y = 0$ to $y = b$

$x = 0$ to $x = a(1 - y/b)$

$$\iint xy\,dx\,dy = \int_0^b \int_0^{a(1-y/b)} xy\,dx\,dy$$

$$= \int_0^b y\left[\frac{x^2}{2}\right]_0^{a(1-y/b)} dy$$

$$= \frac{1}{2}\int_0^b y\left[x^2\right]_0^{a(1-y/b)} dy$$

$$= \frac{1}{2}\int_0^b y\left[a(1-y/b)\right]^2 dy$$

$$= \frac{1}{2}\int_0^b y\, a^2 (1-y/b)^2\, dy$$

$$= \frac{a^2}{2}\int_0^b y(1-y/b)^2\, dy$$

$$= \frac{a^2}{2}\int_0^b y\left(1 + \frac{y^2}{b^2} - 2y/b\right) dy$$

$$= \frac{a^2}{2}\int_0^b \left(y + \frac{y^3}{b^2} - \frac{2y^2}{b}\right) dy$$

$$= \frac{a^2}{2}\left[\frac{y^2}{2} + \frac{y^4}{4b^2} - \frac{2y^3}{3b}\right]_0^b$$

$$= \frac{a^2}{2}\left[\frac{b^2}{2} + \frac{b^{\cancel{4}2}}{4\cancel{b^2}} - \frac{2b^{\cancel{3}2}}{3\cancel{b}}\right] - [0]$$

$$= \frac{a^2}{2}\left[\frac{b^2}{2} + \frac{b^2}{4} - \frac{2b^2}{3}\right]$$

$$= \frac{a^2}{2}\left[\frac{b^2}{2} + \frac{b^2}{4} - \frac{2b^2}{3}\right]$$

$$= \frac{a^2b^2}{2}\left[\frac{1}{2} + \frac{1}{4} - \frac{2}{3}\right]$$

$$= \frac{a^2b^2}{2}\left[\frac{6+3-8}{12}\right] = \frac{a^2b^2}{2}\left[\frac{9-8}{12}\right]$$

$$= \frac{a^2b^2}{2}\left(\frac{1}{12}\right)$$

$$\boxed{I = \frac{a^2b^2}{24}}$$

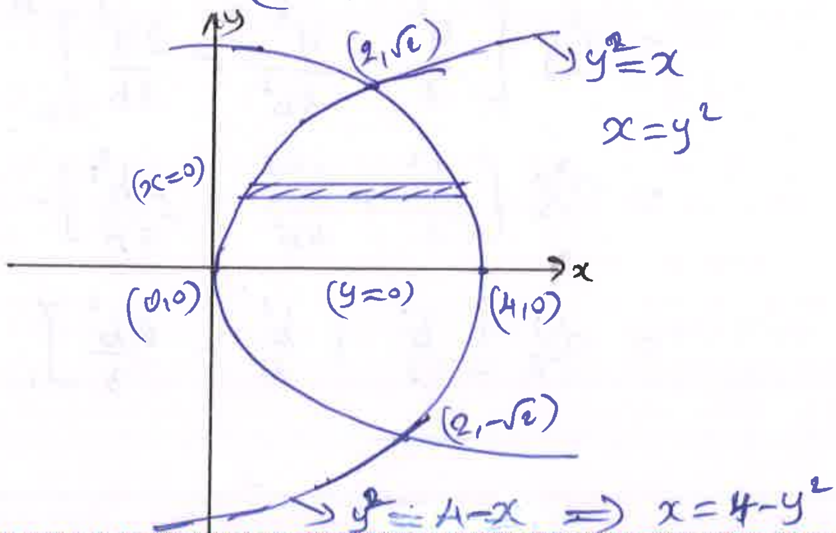
Example-5 Find the area bounded by the parabolas $y^2 = 4-x$ and $y^2 = x$.

Soln.

Given that $\underbrace{y^2 = 4-x}_{①}$ and $y^2 = x \longrightarrow ②$

Here $y^2 = x$ is the form of the parabola $y^2 = 4ax$ with vertex $(0,0)$, since $(y-0)^2 = (x-0)$ with $4a = 1$

and Here $y^2 = (4-x) = -(x-4)$ is of the form, the parabola $(4,0)$, since $(y-0)^2 = -(x-4)$, with $4a = -1$



from ① & ②

② ⟹ $y^2 = 4-x$

$x = 4-x$

$2x = 4$

$x = 4/2$

$\boxed{x=2}$

① ⟹ $y^2 = x$

$y^2 = 2$

$\boxed{y = \pm\sqrt{2}}$

$y = \sqrt{2}, -\sqrt{2}$

Limits :-

$y = 0$ to $y = \sqrt{2}$

$x = y^2$ to $x = 4-y^2$

$$\text{Required Area} = \int_0^{\sqrt{2}}\int_{y^2}^{4-y^2} dx\,dy$$

$$= \int_0^{\sqrt{2}} [x]_{y^2}^{4-y^2} dy$$

$$= \int_0^{\sqrt{2}} [4-y^2-y^2]\,dy$$

$$= \int_0^{\sqrt{2}} (4-2y^2)\,dy$$

$$= \left[4y - \frac{2y^3}{3}\right]_0^{\sqrt{2}}$$

$$= \left[4\sqrt{2} - \frac{2(\sqrt{2})^3}{3}\right] - [0]$$

$$= 4\sqrt{2} - \frac{2(2\sqrt{2})}{3}$$

$$= 4\sqrt{2} - \frac{4\sqrt{2}}{3} = 4\sqrt{2}\left[1 - \frac{1}{3}\right]$$

$$= 4\sqrt{2}\left[\frac{3-1}{3}\right] = 4\sqrt{2}\left(\frac{2}{3}\right) = \frac{8}{3}\sqrt{2}$$

$$\text{Area} = \frac{8\sqrt{2}}{3}$$

Chapter-5.2 [Double Integration in Polar Coordinates]

Example-① Evaluate $\int_0^{\pi}\int_0^{\cos\theta} r\,dr\,d\theta$

Soln:- Given that $\int_0^{\pi}\int_0^{\cos\theta} r\,dr\,d\theta$

$$= \int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{\cos\theta} d\theta = \int_0^{\pi} \frac{\cos^2\theta}{2}\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi} \cos^2\theta\, d\theta = \frac{1}{2}\int_0^{\pi} (1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi}$$

$$= \frac{1}{2}\left\{\left[\pi + \frac{\sin(2\pi)}{2}\right] - \left[0 + \frac{\sin(0)}{2}\right]\right\}$$

$$= \frac{1}{2}\left\{\left[\frac{\pi}{2} + 0\right] - [0+0]\right\}$$

$$= \frac{1}{2}\left(\frac{\pi}{2}\right)$$

$$\boxed{I = \pi/4}$$

Example-② Evaluate $\int_0^{\pi}\int_0^{a\sin\theta} r\,dr\,d\theta$

Soln: Given that $\int_0^{\pi}\int_0^{a\sin\theta} r\,dr\,d\theta$

$$= \int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{a\sin\theta} d\theta = \frac{1}{2}\int_0^{\pi}\left[r^2\right]_0^{a\sin\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi} (a\sin\theta)^2 d\theta = \frac{1}{2}\int_0^{\pi} a^2 \sin^2\theta \, d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi} \sin^2\theta \, d\theta = \frac{a^2}{2}\int_0^{\pi} \left[\frac{1-\cos 2\theta}{2}\right] d\theta$$

$$= \frac{a^2}{4}\int_0^{\pi} [1-\cos 2\theta]\, d\theta$$

$$= \frac{a^2}{4}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi}$$

$$= \frac{a^2}{4}\left\{\left[\pi - \frac{\sin(2\pi)}{2}\right] - \left[0 - \frac{\sin(0)}{2}\right]\right\}$$

$$= \frac{a^2}{4}\left\{[\pi - 0] - (0)\right\}$$

$$= \frac{a^2}{4}(\pi)$$

$$\boxed{I = \frac{\pi a^2}{4}}$$

Example-③ Evaluate $\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^2 \sin\theta \, dr\, d\theta$.

Soln: Given that $\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^2 \sin\theta \, dr\, d\theta$

$$= \int_0^{\pi} \sin\theta \left[\frac{r^3}{3}\right]_0^{a(1+\cos\theta)} d\theta$$

$$= \frac{1}{3}\int_0^{\pi} \sin\theta \left[r^3\right]_0^{a(1+\cos\theta)} d\theta$$

$$= \frac{1}{3}\int_0^{\pi} \sin\theta \left[a(1+\cos\theta)\right]^3 d\theta$$

$$= \frac{1}{3}\int_0^{\pi} \sin\theta \, a^3 (1+\cos\theta)^3 d\theta$$

$$= \frac{a^3}{3}\int_0^{\pi}(1+\cos\theta)^3 \sin\theta\, d\theta$$

Put, $z = 1+\cos\theta$ | $\theta = 0 \Rightarrow z = 1+\cos(0) = 1+1 = 2$

$dz = -\sin\theta\, d\theta$ | $\boxed{z=2}$

$-dz = \sin\theta\, d\theta$ | $\theta = \pi \Rightarrow z = 1+\cos\pi = 1-1 = 0$

$\boxed{z=0}$

$$= \frac{a^3}{3}\int_2^0 z^3(-dz) \qquad \left\{ \text{W.K.T } \int_a^b f(x)dx = -\int_b^a f(x)dx \right\}$$

$$= \frac{a^3}{3}\int_0^2 z^3 dz$$

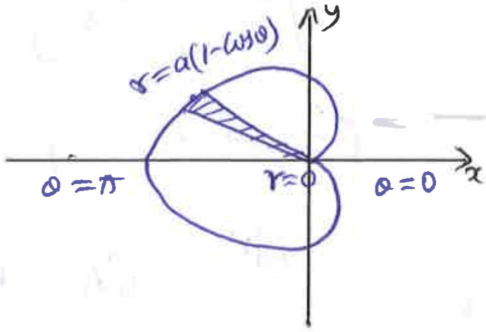
$$= \frac{a^3}{3}\left[\frac{z^4}{4}\right]_0^2 = \frac{a^3}{3}\left[\frac{(2)^4}{4}\right]$$

$$= \frac{a^3}{3}\left(\frac{\cancel{16}^{4}}{\cancel{4}}\right) = \frac{a^3}{3}(4)$$

$$\boxed{I = \frac{4a^3}{3}}$$

Example-④ Evaluate $\iint r\sin\theta\, dr\, d\theta$ over the cardioid $r = a(1-\cos\theta)$ above the initial line.

Soln:- Given that cardioid $r = a(1-\cos\theta)$



Limits:

$r = 0$ to $r = a(1-\cos\theta)$

$\theta = 0$ to $\theta = \pi$

$$\iint r\sin\theta\, dr\, d\theta = \int_0^{\pi}\int_0^{a(1-\cos\theta)} r\sin\theta\, dr\, d\theta$$

$$= \int_0^{\pi} \sin\theta \left[\frac{r^2}{2}\right]_0^{a(1-\cos\theta)} d\theta$$

$$= \frac{1}{2}\int_0^{\pi} \sin\theta \left[r^2\right]_0^{a(1-\cos\theta)} d\theta$$

$$= \frac{1}{2}\int_0^{\pi} \sin\theta \left[a(1-\cos\theta)\right]^2 d\theta$$

$$= \frac{1}{2}\int_0^{\pi} \sin\theta \; a^2 (1-\cos\theta)^2 d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi} (1-\cos\theta)^2 \sin\theta \, d\theta$$

Put, $z = 1-\cos\theta$ | $\theta = 0 \Rightarrow z = 1-\cos(0) = 1-1 = 0$

$dz = \sin\theta \, d\theta$ | $\boxed{z=0}$

$\theta = \pi \Rightarrow z = 1-\cos(\pi) = 1-(-1) = 1+1$

$\boxed{z=2}$

$$= \frac{a^2}{2}\int_0^{2} z^2 \, dz$$

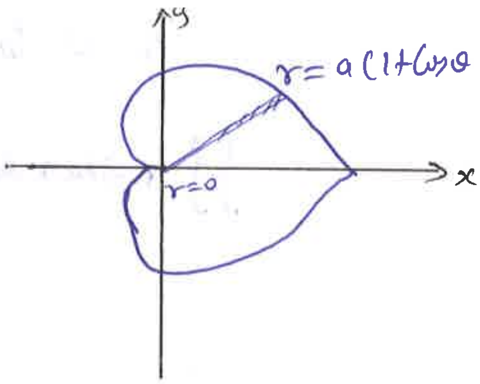
$$= \frac{a^2}{2}\left[\frac{z^3}{3}\right]_0^2 = \frac{a^2}{6}\left[z^3\right]_0^2 = \frac{a^2}{6}\left[(2)^3-(0)\right]$$

$$= \frac{a^2}{\cancel{6}_3}(\cancel{8})^4 = \frac{a^2}{3}(4)$$

$$\boxed{I = \frac{4a^2}{3}}$$

Example-⑤ Find the area of the cardioid $r = a(1+\cos\theta)$.

Soln: Given that $r = a(1+\cos\theta)$



Limits: $r=0$ to $r=a(1+\cos\theta)$

$\theta = -\pi$ to $\theta = \pi$

The Required area $= \iint r\,dr\,d\theta$

$$= \int_{-\pi}^{\pi}\int_{0}^{a(1+\cos\theta)} r\,dr\,d\theta$$

$$= \int_{-\pi}^{\pi}\left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta = \frac{1}{2}\int_{-\pi}^{\pi}\left[r^2\right]_0^{a(1+\cos\theta)} d\theta$$

$$= \frac{1}{2}\int_{-\pi}^{\pi}\left[a(1+\cos\theta)\right]^2 d\theta = \frac{1}{2}\int_{-\pi}^{\pi} a^2(1+\cos\theta)^2 d\theta$$

$$= \frac{a^2}{2}\int_{-\pi}^{\pi}(1+\cos\theta)^2 d\theta \qquad \left\{\begin{array}{l}\text{W.K.T}\\ 1+\cos\theta = 2\cos^2(\theta/2)\end{array}\right.$$

$$= \frac{a^2}{2}\int_{-\pi}^{\pi}\left[2\cos^2(\theta/2)\right]^2 d\theta$$

$$= \frac{a^2}{2}\int_{-\pi}^{\pi} 4\cos^4(\theta/2)\,d\theta = \frac{a^2}{\cancel{2}}\,\cancel{2}\int_{0}^{\pi} 4\cos^4(\theta/2)\,d\theta$$

$$= a^2 4\int_{0}^{\pi}\cos^4(\theta/2)\,d\theta$$

Put, $z = \theta/2$ | $\theta = 0 \Rightarrow z = 0/2 = 0$ $\boxed{z=0}$

$2z = \theta$ | $\theta = \pi \Rightarrow \boxed{z = \pi/2}$

$2\,dz = d\theta$

$$= 4a^2\int_{0}^{\pi/2}\cos^4(z)\,(2\,dz) = 8a^2\int_{0}^{\pi/2}\cos^4(z)\,dz.$$

$\left\{ \text{W.K.T} \int_0^{\pi}\cos^n\theta\,d\theta = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{1}{2}\cdot\frac{\pi}{2} \right\}$ n is Even.

$$= 8a^2\left[\frac{4-1}{4}\cdot\frac{4-3}{4-2}\cdot\frac{\pi}{2}\right]$$

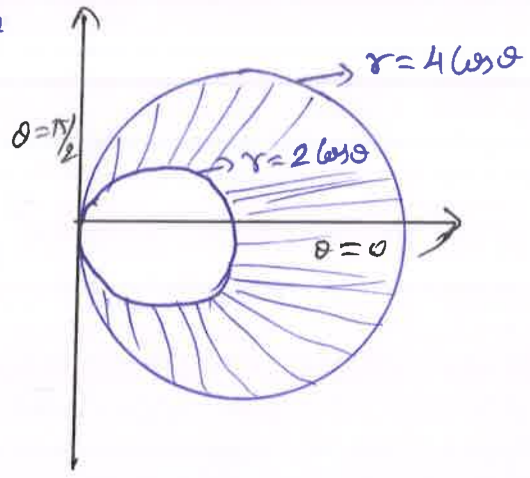
$$= 8a^2\left[\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right] = \cancel{8}a^2\left[\frac{3\pi}{\cancel{16}_{2}}\right]$$

Area. $\boxed{I = \frac{3\pi a^2}{2}}$

Example-6 Find the area of the region outside the inner circle $r = 2\cos\theta$ and inside the outer circle $r = 4\cos\theta$.

Sol/n: Given that $r = 2\cos\theta$ and $r = 4\cos\theta$

$\theta = 0$ to $\theta = \pi/2$



$$\text{Required Area} = 2\iint r\,dr\,d\theta$$

$$= 2\int_0^{\pi/2}\int_{2\cos\theta}^{4\cos\theta} r\,dr\,d\theta$$

$$= \cancel{2}\int_0^{\pi/2}\left[\frac{r^2}{\cancel{2}}\right]_{2\cos\theta}^{4\cos\theta} d\theta = \int_0^{\pi/2}\left[r^2\right]_{2\cos\theta}^{4\cos\theta} d\theta$$

$$= \int_0^{\pi/2}\left[(4\cos\theta)^2 - (2\cos\theta)^2\right] d\theta$$

$$= \int_0^{\pi/2}(16\cos^2\theta - 4\cos^2\theta)\,d\theta = \int_0^{\pi/2} 12\cos^2\theta\,d\theta$$

$$= 12\int_0^{\pi/2}\cos^2\theta\,d\theta$$

$$= 12\left[\tfrac{1}{2}\cdot\tfrac{\pi}{2}\right] = \overset{3}{\cancel{12}}\left[\tfrac{\pi}{\cancel{4}}\right]$$

$$\boxed{I = 3\pi}$$

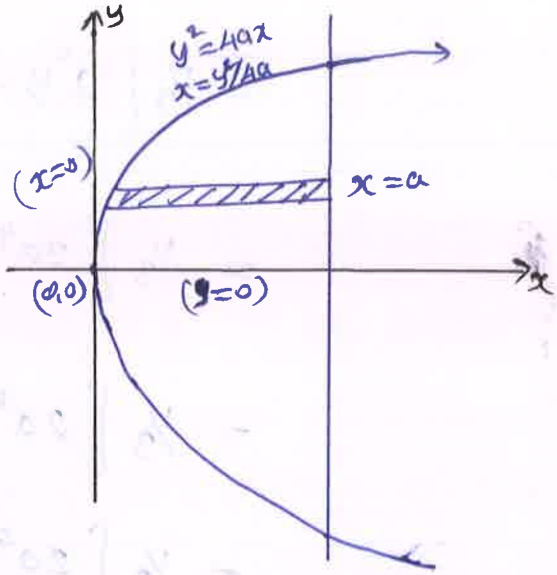
Chapter - 5.3 [Change the order of Integration]

Note: $\iint f(x,y)\,dx\,dy = \iint f(x,y)\,dy\,dx$

Example - 1

Change the order of integration in the integral

$\int_0^a \int_0^{2\sqrt{ax}} x^2\,dy\,dx$ and Evaluate.



Soln: Given that $\int_0^a \int_0^{2\sqrt{ax}} x^2\,dy\,dx$

Given Limit: $x=0$ to $x=a$

$y=0$ to $y=2\sqrt{ax}$

$y^2=4ax$

Given order $= dy\,dx$

To change the order $= dx\,dy$

$y^2=4ax$

Put $\boxed{x=a}$

$y^2=4a(a)$

$y^2=4a^2$

$\boxed{y=2a}$

Changed Limit: $y=0$ to $y=2a$

$x=\frac{y^2}{4a}$ to $x=a$

$$\int_0^a \int_0^{2\sqrt{ax}} x^2\,dy\,dx = \int_0^{2a}\int_{\frac{y^2}{4a}}^{a} x^2\,dx\,dy$$

$$= \int_0^{2a}\left[\frac{x^3}{3}\right]_{\frac{y^2}{4a}}^{a} dy = \frac{1}{3}\int_0^{2a}\left[x^3\right]_{\frac{y^2}{4a}}^{a} dy$$

$$= \frac{1}{3}\int_0^{2a}\left[a^3 - \left(\frac{y^2}{4a}\right)^3\right].dy$$

$$= \frac{1}{3}\int_0^{2a}\left[a^3 - \frac{y^6}{(4a)^3}\right]dy$$

$$= \frac{1}{3}\int_0^{2a}\left[a^3 - \frac{y^6}{64a^3}\right]dy$$

$$= \frac{1}{3}\left[a^3y - \frac{y^7}{7\times 64a^3}\right]_0^{2a}$$

$$= \frac{1}{3}\left[a^3y - \frac{y^7}{448a^3}\right]_0^{2a} = \frac{1}{3}\left[a^3(2a) - \frac{(2a)^7}{448a^3}\right]$$

$$= \frac{1}{3}\left[2a^4 - \frac{128a^{\cancel{7}\,4}}{448\cancel{a^3}}\right] = \frac{1}{3}\left[2a^4 - \frac{128a^4}{448}\right]$$

$$= \frac{1}{3}\left[2a^4 - \frac{16a^4}{56}\right] = \frac{1}{3}\left[2a^4 - \frac{4a^4}{14}\right]$$

$$= \frac{1}{3}\left[2a^4 - \frac{2a^4}{7}\right] = \frac{1}{3}\left[\frac{14a^4 - 2a^4}{7}\right]$$

$$= \frac{1}{\cancel{3}}\left[\frac{\cancel{12}^{\,4}a^4}{7}\right] = \frac{4a^4}{7}$$

$$\boxed{I = \frac{4a^4}{7}}$$

Example-② Change the order of integration in $\int_0^a\int_x^a (x^2+y^2)\,dy\,dx$ and hence evaluate it.

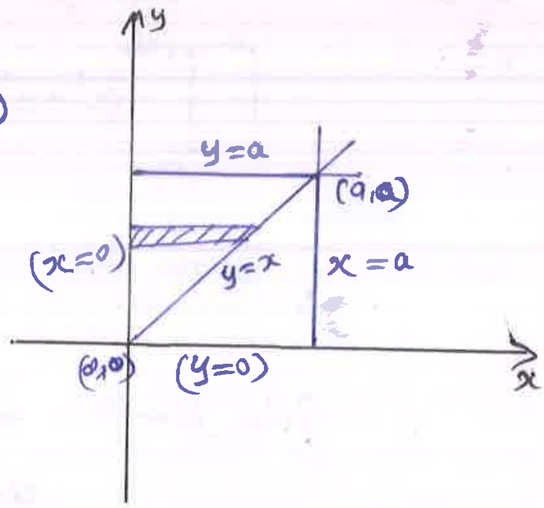
Soln:- Given that $\int_0^a\int_x^a (x^2+y^2)\,dy\,dx$.

Given Limit:- $x = 0$ to $x = a$

$y = x$ to $y = a$

Given order $= dy\,dx$.

To change the order $= dx\,dy$



• Changed Limit :-

$y=0$ to $y=a$

$x=0$ to $x=y$

$$\int_0^a\int_x^a (x^2+y^2)\,dy\,dx = \int_0^a\int_0^y (x^2+y^2)\,dx\,dy$$

$$= \int_0^a \left[\frac{x^3}{3}+y^2x\right]_0^y dy$$

$$= \int_0^a \left[\frac{y^3}{3}+y^2(y)\right]-[0]\;dy$$

$$= \int_0^a \left[\frac{y^3}{3}+y^3\right] dy$$

$$= \left[\frac{y^4}{12}+\frac{y^4}{4}\right]_0^a$$

$$= \left[\frac{a^4}{12}+\frac{a^4}{4}\right]$$

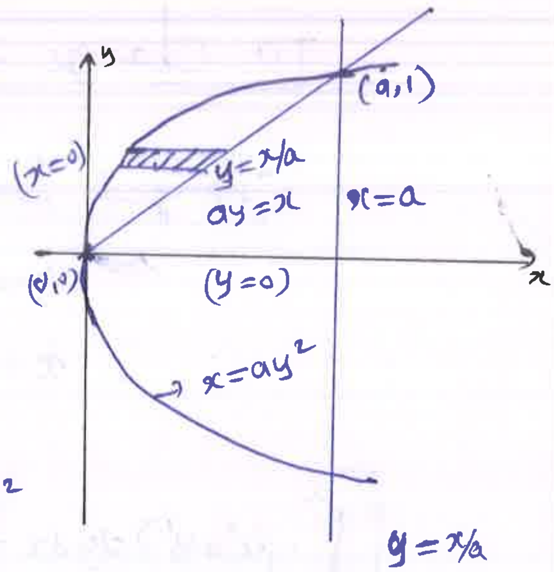
$$= \frac{a^4+3a^4}{12}$$

$$= \frac{4a^4}{12_3}$$

$$\boxed{I = \frac{a^4}{3}}$$

Example-(3) Change the order of Integration and Evaluate

$$\int_0^a \int_{x/a}^{\sqrt{x/a}} (x^2+y^2)\,dy\,dx$$



Sol/n:

Given that $\int_0^a \int_{x/a}^{\sqrt{x/a}} (x^2+y^2)\,dy\,dx$

Given Limit:- $x=0$ to $x=a$

$y=x/a$ to $y=\sqrt{x/a}$

$y^2 = x/a \Rightarrow x=ay^2$

$y=x/a$

$y=a/a$

$\boxed{y=1}$

Given order $= dy\,dx$

To change the order $\boxed{= dx\,dy}$

Changed Limit:- $y=0$ to $y=1$

$x=ay^2$ to $x=ay$

$$\int_0^a \int_{x/a}^{\sqrt{x/a}} (x^2+y^2)\,dy\,dx = \int_0^1 \int_{ay^2}^{ay} (x^2+y^2)\,dx\,dy$$

$$= \int_0^1 \left[\frac{x^3}{3} + y^2 x\right]_{ay^2}^{ay} dy$$

$$= \int_0^1 \left[\frac{(ay)^3}{3} + y^2(ay)\right] - \left[\frac{(ay^2)^3}{3} + y^2(ay^2)\right] dy$$

$$= \int_0^1 \left[\frac{a^3y^3}{3} + ay^3 - \frac{a^3y^6}{3} - ay^4\right] dy$$

$$= \left[\frac{a^3y^4}{12} + \frac{ay^4}{4} - \frac{a^3y^7}{21} - \frac{ay^5}{5}\right]_0^1$$

$$= \left[\frac{a^3(1)}{12} + \frac{a(1)}{4} - \frac{a^3(1)}{21} - \frac{a(1)}{5}\right] - [0]$$

$$= \left[\frac{a^3}{12} + \frac{a}{4} - \frac{a^3}{21} - \frac{a}{5} \right]$$

$$= \left(\frac{a^3}{12} - \frac{a^3}{21} \right) + \left(\frac{a}{4} - \frac{a}{5} \right)$$

$$= \left(\frac{7a^3 - 4a^3}{84} \right) + \left(\frac{5a - 4a}{20} \right)$$

$$= \frac{3a^3}{84} + \frac{a}{20}$$

$$\boxed{I = \frac{a^3}{28} + \frac{a}{20}}$$

Example-5 Change the order of integration and hence Evaluate the integral $\int_0^a \int_{x^2/a}^{2a-x} xy \, dy \, dx$.

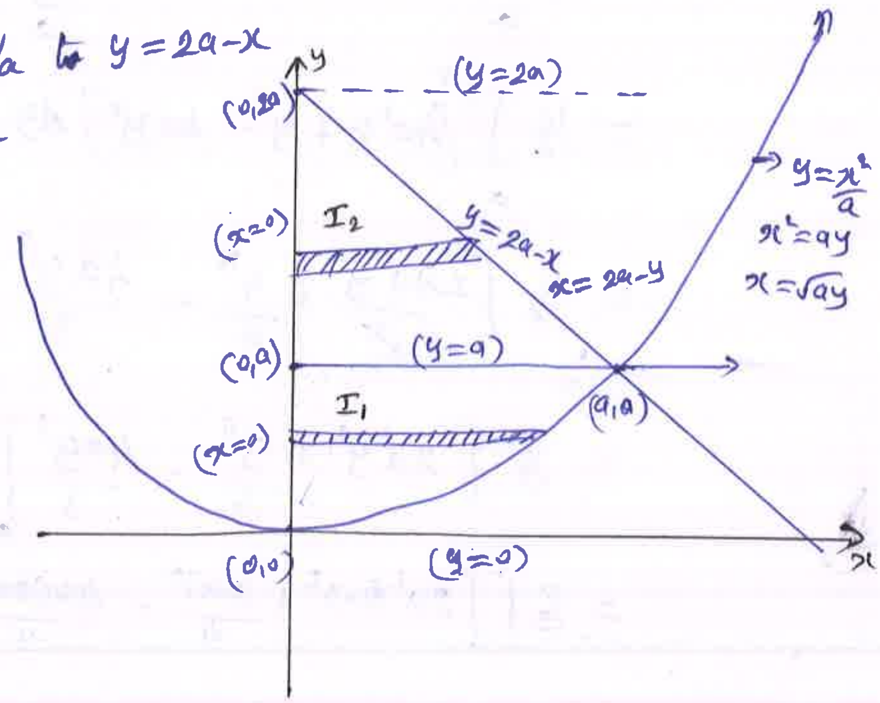
Soln: Given that $\int_0^a \int_{x^2/a}^{2a-x} xy \, dy \, dx$

Given order $= dy\,dx$

Given Limits: $x = 0$ to $x = a$

$y = x^2/a$ to $y = 2a - x$

$ay = x^2$



To change the order $\boxed{= dx\,dy}$

Changed Limit for I_1	Changed Limit for I_2
$y=0$ to $y=a$	$y=a$ to $y=2a$
$x=0$ to $x=\sqrt{ay}$	$x=0$ to $x=2a-y$

$$I_1 = \int_0^a\int_0^{\sqrt{ay}} xy\,dx\,dy = \int_0^a \left[\frac{yx^2}{2}\right]_0^{\sqrt{ay}} dy = \frac{1}{2}\int_0^a \left[yx^2\right]_0^{\sqrt{ay}} dy$$

$$= \frac{1}{2}\int_0^a y(\sqrt{ay})^2 dy = \frac{1}{2}\int_0^a y(ay)\,dy = \frac{1}{2}\int_0^a ay^2\,dy$$

$$= \frac{1}{2}\left[\frac{ay^3}{3}\right]_0^a = \frac{1}{2}\left[\frac{a(a^3)}{3}\right] = \frac{1}{2}\left(\frac{a^4}{3}\right)$$

$$\boxed{I_1 = \frac{a^4}{6}}$$

$$I_2 = \int_a^{2a}\int_0^{2a-y} xy\,dx\,dy = \int_a^{2a}\left[\frac{yx^2}{2}\right]_0^{2a-y} dy = \frac{1}{2}\int_a^{2a}\left[yx^2\right]_0^{2a-y} dy$$

$$= \frac{1}{2}\int_a^{2a}\left[y(2a-y)^2\right] dy = \frac{1}{2}\int_a^{2a} y\left[4a^2+y^2-4ay\right] dy$$

$$= \frac{1}{2}\int_a^{2a}\left[4a^2y+y^3-4ay^2\right] dy$$

$$= \frac{1}{2}\left[\frac{\cancel{4}^{2}a^2y^2}{\cancel{2}}+\frac{y^4}{4}-\frac{4ay^3}{3}\right]_a^{2a}$$

$$= \frac{1}{2}\left[2a^2y^2+\frac{y^4}{4}-\frac{4ay^3}{3}\right]_a^{2a}$$

$$= \frac{1}{2}\left\{\left[2a^2(2a)^2+\frac{(2a)^4}{4}-\frac{4a(2a)^3}{3}\right]-\left[2a^2(a^2)+\frac{a^4}{4}-\frac{4a(a^3)}{3}\right]\right\}$$

$$= \frac{1}{2}\left\{\left[2a^2(4a^2) + \frac{16a^4}{4} - \frac{4a(8a^3)}{3}\right] - \left[2a^4 + \frac{a^4}{4} - \frac{4a^4}{3}\right]\right\}$$

$$= \frac{1}{2}\left\{\left[8a^4 + 4a^4 - \frac{32a^4}{3}\right] - \left[2a^4 + \frac{a^4}{4} - \frac{4a^4}{3}\right]\right\}$$

$$= \frac{1}{2}\left\{12a^4 - \frac{32a^4}{3} - 2a^4 - \frac{a^4}{4} + \frac{4a^4}{3}\right\}$$

$$= \frac{1}{2}\left\{10a^4 - \frac{a^4}{4} - \frac{28a^4}{3}\right\} = \frac{a^4}{2}\left[10 - \frac{1}{4} - \frac{28}{3}\right]$$

$$= \frac{a^4}{2}\left[\frac{120 - 3 - 112}{12}\right] = \frac{a^4}{2}\left[\frac{120 - 115}{12}\right]$$

$$= \frac{a^4}{2}\left[\frac{5}{12}\right]$$

$$\boxed{I_2 = \frac{5a^4}{24}}$$

$$\therefore I = I_1 + I_2$$

$$= \frac{a^4}{6} + \frac{5a^4}{24}$$

$$= \frac{4a^4 + 5a^4}{24}$$

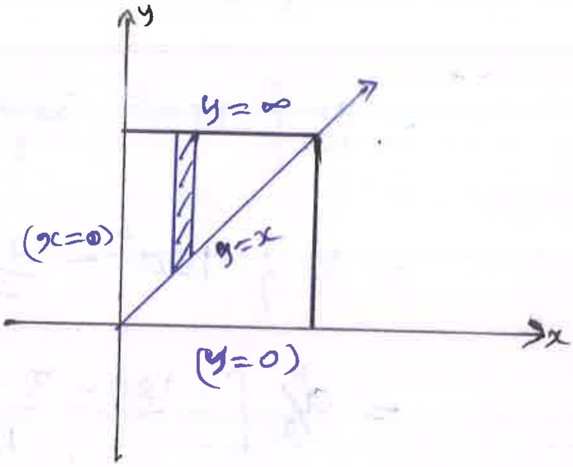
$$= \frac{9a^4}{24}$$

$$\boxed{I = \frac{3a^4}{8}}$$

Example-6 Change the order of integration $\int_0^\infty \int_0^y y e^{-y^2/x} \, dx \, dy$.

Soln:

Given that $\int_0^\infty \int_0^y y e^{-y^2/x} \, dx \, dy$



Given Limit:- $y = 0$ to $y = \infty$

$x = 0$ to $x = y$

Given order $= dx\,dy$

To change the order $\boxed{= dy\,dx}$

Changed Limit:- $x = 0$ to $x = \infty$

$y = x$ to $y = \infty$

$$\int_0^\infty \int_0^y y e^{-y^2/x} \, dx \, dy = \int_0^\infty \int_x^\infty y \, e^{-y^2/x} \, dy \, dx$$

$$= \frac{1}{2} \int_0^\infty \int_x^\infty 2y \, e^{-y^2/x} \, dy\,dx = \frac{1}{2} \int_0^\infty \int_x^\infty y \, e^{-y^2/x} \, d(y^2) \, dx$$

$$= \frac{1}{2} \int_0^\infty \left[\frac{e^{-y^2/x}}{-1/x} \right]_x^\infty dx = \frac{1}{2} \int_0^\infty \left[-x \, e^{-y^2/x} \right]_x^\infty dx$$

$$= \frac{1}{2} \int_0^\infty \left[0 - (-x \, e^{-x^2/x}) \right] dx = \frac{1}{2} \int_0^\infty x \, e^{-x} \, dx$$

$$= \frac{1}{2} \left[x \frac{e^{-x}}{-1} - (1) \frac{e^{-x}}{1} \right]_0^\infty$$

$$= \frac{1}{2} \left[-x e^{-x} - e^{-x} \right]_0^\infty$$

$$= \frac{1}{2} \left\{ \left[e^{-\infty} - e^{-\infty} \right] - \left[(0) e^{0} - e^{0} \right] \right\}$$

$$\left\{ \begin{array}{l} \text{W.K.T} \\ e^{-\infty} = 0 \\ e^{0} = 1 \end{array} \right\}$$

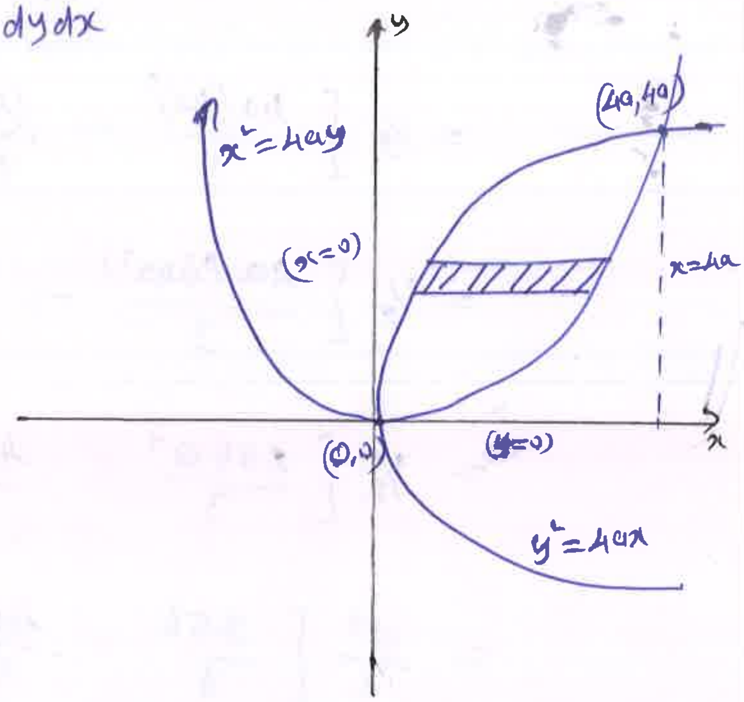
$$= \frac{1}{2} \left[0 - (-1) \right] = \frac{1}{2}(1)$$

$$\boxed{I = \frac{1}{2}}$$

Example-(7) Change the order of Integration and hence Evaluate it

$$\int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} xy\,dy\,dx.$$

Soln: Given that $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} xy\,dy\,dx$



Given Limit: $x=0$ to $x=4a$

$y=\frac{x^2}{4a}$ to $y=2\sqrt{ax}$

$4ay=x^2$ to $y^2=4ax$

Given order $= dy\,dx$

To change the order $\boxed{= dx\,dy}$

Changed Limit: $y=0$ to $y=4a$

$x=\frac{y^2}{4a}$ to $x=2\sqrt{ay}$

$$\int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} xy\,dy\,dx = \int_0^{4a}\int_{\frac{y^2}{4a}}^{2\sqrt{ay}} xy\,dx\,dy$$

$$= \int_0^{4a}\left[\frac{yx^2}{2}\right]_{y^2/4a}^{2\sqrt{ay}} dy = \frac{1}{2}\int_0^{4a}\left[yx^2\right]_{y^2/4a}^{2\sqrt{ay}} dy$$

$$= \frac{1}{2}\int_0^{4a}\left[y(2\sqrt{ay})^2 - y\left(\frac{y^2}{4a}\right)^2\right] dy$$

$$= \frac{1}{2}\int_0^{4a}\left[y(4ay) - y\left(\frac{y^4}{16a^2}\right)\right] dy$$

$$= \frac{1}{2}\int_0^{4a}\left[4ay^2 - \frac{y^5}{16a^2}\right] dy$$

$$= \frac{1}{2}\left[\frac{4ay^3}{3} - \frac{y^6}{6\times 16a^2}\right]_0^{4a}$$

$$= \frac{1}{2}\left[\frac{4ay^3}{3} - \frac{y^6}{96a^2}\right]_0^{4a}$$

$$= \frac{1}{2}\left[\frac{4a(4a)^3}{3} - \frac{(4a)^6}{96a^2}\right] - [0]$$

$$= \frac{1}{2}\left[\frac{4a(64a^3)}{3} - \frac{4096\,a^{\cancel{6}\,4}}{96\,\cancel{a^2}}\right]$$

$$= \frac{1}{2}\left[\frac{256a^4}{3} - \frac{4096a^4}{96}\right]$$

$$= \frac{a^4}{2}\left[\frac{256}{3} - \frac{4096}{96}\right]$$

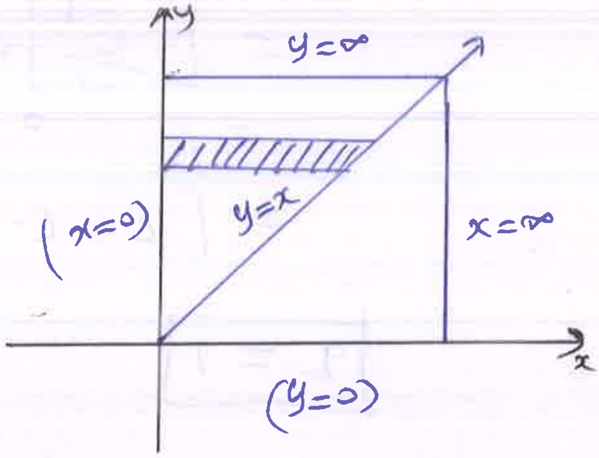
$$= \frac{a^4}{2}\left[\frac{32(256) - 4096}{96}\right]$$

$$= \frac{a^4}{2}\left[\frac{8192 - 4096}{96}\right] = \frac{a^4}{2}\left[\frac{\overset{512}{\cancel{4096}}}{\underset{12}{\cancel{96}}}\right] = \frac{a^4}{2}\left[\frac{512}{12}\right]$$

$$= \frac{a^4}{\cancel{2}}\left[\frac{\overset{64}{\cancel{128}}}{3}\right] = a^4\left(\frac{64}{3}\right)$$

$$\boxed{I = \frac{64a^4}{3}}$$

Example-⑧ change the order of integration in $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx$ and then Evaluate.



Soln:- Given that $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx$

Given limits:-

$x=0$ to $x=\infty$

$y=x$ to $y=\infty$

Given order = $dy\,dx$

To change the order $\boxed{= dx\,dy}$

Changed limit:- $y=0$ to $y=\infty$

$x=0$ to $x=y$

$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx = \int_0^\infty \int_0^y \frac{e^{-y}}{y}\, dx\, dy$$

$$= \int_0^\infty \left(\frac{e^{-y}}{y}\right) [x]_0^y\, dy$$

$$= \int_0^\infty \frac{e^{-y}}{y} [y-0]\, dy$$

$$= \int_0^\infty \frac{e^{-y}}{\cancel{y}} (\cancel{y})\, dy$$

$$= \int_0^\infty e^{-y}\, dy$$

$$= \int_0^\infty e^{-y}\,dy$$

$$= \left[\frac{e^{-y}}{-1}\right]_0^\infty = \left[-e^{-y}\right]_0^\infty = \left[-e^{-\infty} - (-e^0)\right]$$

$$= [0 - (-1)] = 1$$

$$\boxed{I = 1}$$

Chapter-~~5~~.4 [Change into Polar Co-ordinates]

Note:-

Let $x = r\cos\theta$; $y = r\sin\theta$

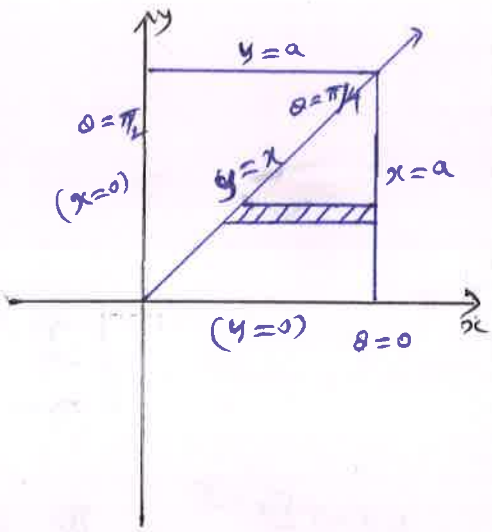
$r^2 = x^2 + y^2$

$dx\,dy = dy\,dx = r\,dr\,d\theta$

$\sin(0) = 0$
$\cos(0) = 1$
$\sin n\pi = 0$
$\cos n\pi = (-1)^n$

Example-① Evaluate by changing to polar co-ordinates

$$\int_0^a \int_y^a \frac{x}{x^2+y^2}\,dx\,dy.$$



Soln Given that $\int_0^a \int_y^a \frac{x}{x^2+y^2}\,dx\,dy$

Given Limit: $y = 0$ to $y = a$

$x = y$ to $x = a$

To change into Polar Co-ordinates

Let $x = r\cos\theta$; $y = r\sin\theta$

$x^2 + y^2 = r^2$; $dx\,dy = r\,dr\,d\theta$

Here $\boxed{r = 0}$; $x = a \Rightarrow r\cos\theta = a$

$r = \frac{a}{\cos\theta} = a\sec\theta$

$\boxed{r = a\sec\theta}$

$\left\{ \begin{array}{l} \text{W.K.T} \\ \frac{1}{\cos\theta} = \sec\theta \end{array} \right.$

Limit:

$\theta = 0$ to $\theta = \pi/4$

$r = 0$ to $r = a\sec\theta$

$$\int_0^a \int_y^a \frac{x}{x^2+y^2}\,dx\,dy = \int_0^{\pi/4}\int_0^{a\sec\theta} \frac{\cancel{r}\cos\theta}{r^{\cancel{2}}}\, r\,dr\,d\theta$$

$$= \int_0^{\pi/4}\int_0^{a\sec\theta} \cos\theta\,dr\,d\theta$$

$$= \int_0^{\pi/4} \cos\theta\,[r]_0^{a\sec\theta}\,d\theta$$

$$= \int_0^{\pi/4} \cos\theta\,(a\sec\theta)\,d\theta$$

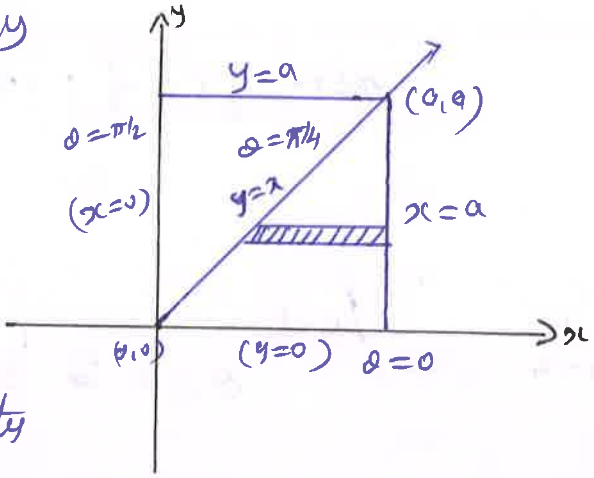
$$= \int_0^{\pi/4} a\,\cancel{\cos\theta}\cdot\frac{1}{\cancel{\cos\theta}}\,d\theta$$

$$= a\int_0^{\pi/4} d\theta = a\,[\theta]_0^{\pi/4} = a\,[\pi/4 - 0]$$

$$\boxed{I = \frac{a\pi}{4}}$$

<u>Example-②</u> By changing to polar co-ordinates, find the value of $\int_0^a\int_y^a \frac{x^2}{\sqrt{x^2+y^2}}\,dx\,dy.$

<u>Soln:</u> Given that $\int_0^a\int_y^a \frac{x^2}{\sqrt{x^2+y^2}}\,dx\,dy$



<u>Given Limits</u>

$y=0$ to $y=a$

$x=y$ to $x=a$

<u>To change into polar co-ordinates</u>

Let $x = r\cos\theta$; $y = r\sin\theta$

$r^2 = x^2+y^2$; $dx\,dy = r\,dr\,d\theta$

Here, $\boxed{r=0}$ and $x=a \Rightarrow r\cos\theta = a$

$r = a/\cos\theta$

$\boxed{r = a\sec\theta}$

Limit:

$\theta = 0$ to $\theta = \pi/4$

$r = 0$ to $r = a\sec\theta$

$$\int_0^a \int_y^a \frac{x^2}{\sqrt{x^2+y^2}}\,dx\,dy = \int_0^{\pi/4}\int_0^{a\sec\theta} \frac{r^2\cos^2\theta}{\sqrt{r^2}}\, r\,dr\,d\theta$$

$$= \int_0^{\pi/4}\int_0^{a\sec\theta} \frac{r^2\cos^2\theta}{\cancel{r}}\, \cancel{r}\,dr\,d\theta$$

$$= \int_0^{\pi/4}\int_0^{a\sec\theta} r^2\cos^2\theta\,dr\,d\theta$$

$$= \int_0^{\pi/4} \cos^2\theta \left[\frac{r^3}{3}\right]_0^{a\sec\theta} d\theta$$

$$= \frac{1}{3}\int_0^{\pi/4} \cos^2\theta \left[r^3\right]_0^{a\sec\theta} d\theta$$

$$= \frac{1}{3}\int_0^{\pi/4} \cos^2\theta \left[a^3\sec^3\theta\right] d\theta$$

$$= \frac{a^3}{3}\int_0^{\pi/4} \cos^2\theta\,\sec^3\theta\,d\theta$$

$$= \frac{a^3}{3}\int_0^{\pi/4} \frac{1}{\cancel{\sec^2\theta}}\,\sec^{\cancel{3}}\theta\,d\theta$$

$$= \frac{a^3}{3}\int_0^{\pi/4} \sec\theta\,d\theta = \frac{a^3}{3}\left[\log(\sec\theta + \tan\theta)\right]_0^{\pi/4}$$

$$= \frac{a^3}{3}\left[\log(\sec\pi/4 + \tan\pi/4) - \log(\sec(0) + \tan(0))\right]$$

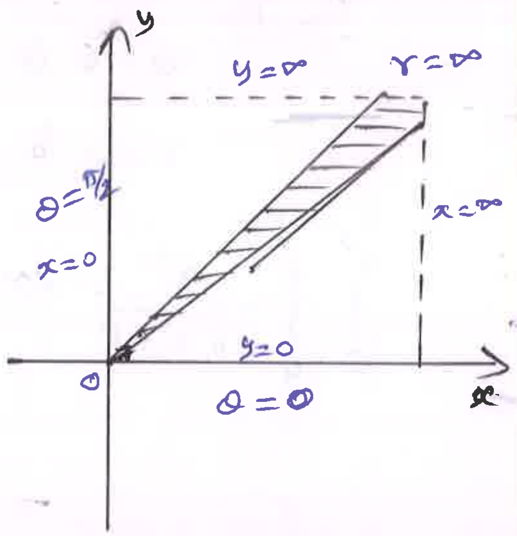
$$= \frac{a^3}{3}\left[\log(\sqrt{2}+1) - 0\right]$$

$$I = \frac{a^3}{3}\log(\sqrt{2}+1)$$ //

Example (3) Evaluate $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$ by Polar Co-ordinates.

<u>Soln:-</u>

Given that $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$



<u>Given Limits.</u> $x=0$ to $x=\infty$

$y=0$ to $y=\infty$

<u>To change into Polar Coordinates</u>

Let $x = r\cos\theta$; $y = r\sin\theta$

$r^2 = x^2+y^2$; $dx\,dy = r\,dr\,d\theta$

<u>Limits:</u> $\theta = 0$ to $\theta = \pi/2$

$r = 0$ to $r = \infty$

$$\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = \int_0^{\pi/2}\int_0^\infty e^{-r^2}\,r\,dr\,d\theta$$

Put, $t = r^2$ | $r=0 \Rightarrow t=0$

$dt = 2r\,dr$ | $r=\infty \Rightarrow t=\infty$

$\frac{dt}{2} = r\,dr$

$$= \int_0^{\pi/2}\int_0^\infty e^{-t}\,\frac{dt}{2}\,d\theta = \frac{1}{2}\int_0^{\pi/2}\int_0^\infty \frac{e^{-t}}{-1}\,dt\,d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2}\left[e^{-t}\right]_0^\infty d\theta = -\frac{1}{2}\int_0^{\pi/2}\left[e^{-\infty}-e^{0}\right]d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2}[0-1]\,d\theta = \frac{1}{2}\int_0^{\pi/2}d\theta$$

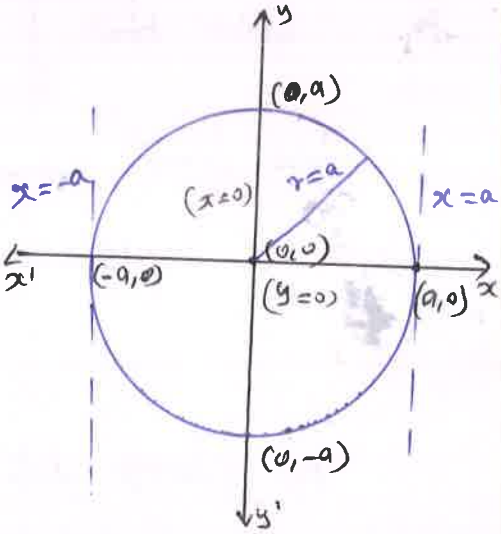
$$= \frac{1}{2}\left[\theta\right]_0^{\pi/2} = \frac{1}{2}\left[\pi/2 - 0\right] = \frac{1}{2}(\pi/2)$$

$$\boxed{I = \pi/4}$$

Example - 4 Change into Polar-Coordinates $\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\,dx$.

soln:-

Given that $\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\,dx$



Given Limits: $x=-a$ to $x=a$

$y=-\sqrt{a^2-x^2}$ to $y=\sqrt{a^2-x^2}$

ie, $\boxed{x=\pm a}$; $y=\pm\sqrt{a^2-x^2}$

$y^2=a^2-x^2 \Rightarrow \boxed{x^2+y^2=a^2}$

ie, $r^2=a^2$

$\boxed{r=a}$

To change into Polar Coordinates

Let $x=r\cos\theta$; $y=r\sin\theta$

$x^2+y^2=r^2$; $dx\,dy=r\,dr\,d\theta$

Limits:

$\theta=0$ to $\theta=2\pi$

$r=0$ to $r=a$

$$\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\,dx=\int_{0}^{2\pi}\int_{0}^{a} r\,dr\,d\theta$$

$$=\int_{0}^{2\pi}\left[\frac{r^2}{2}\right]_{0}^{a}d\theta=\frac{1}{2}\int_{0}^{2\pi}[r^2]_{0}^{a}d\theta=\frac{1}{2}\int_{0}^{2\pi}a^2\,d\theta$$

$$=\frac{a^2}{2}\int_{0}^{2\pi}d\theta=\frac{a^2}{2}\left[\theta\right]_{0}^{2\pi}$$

$$=\frac{a^2}{2}\left[2\pi-0\right]$$

$$=\frac{a^2}{\cancel{2}}(\cancel{2}\pi)$$

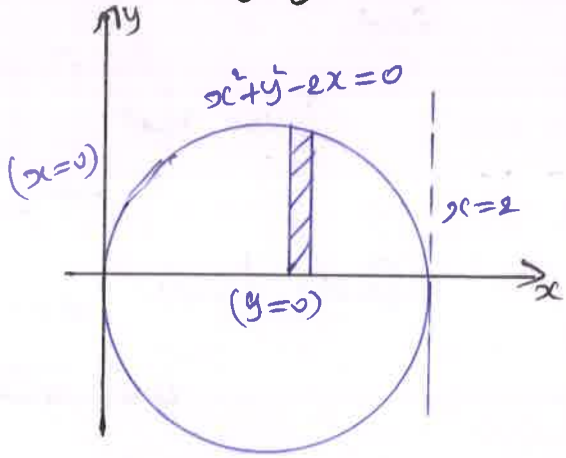
$$=a^2\pi$$

$$\boxed{I=\pi a^2}$$

Example-5 By changing into Polar-Coordinates Evaluate $\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{x^2+y^2}\, dy\, dx$.

Soln:

Given that $\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{x^2+y^2}\, dy\, dx$.



Given Limit: $x=0$ to $x=2$

$y=0$ to $y=\sqrt{2x-x^2}$

$y^2 = 2x-x^2$

$x^2+y^2-2x=0$

To change into polar coordinates

Let $x=r\cos\theta$; $y=r\sin\theta$

$x^2+y^2=r^2$; $dy\,dx = r\,dr\,d\theta$

$x^2+y^2-2x=0$

$r^2-2r\cos\theta=0$

$r(r-2\cos\theta)=0$

$\boxed{r=0}$; $r-2\cos\theta=0$

$\boxed{r=2\cos\theta}$

Limit: $r=0$; $r=2\cos\theta$

$\theta=0$; $\theta=\pi/2$

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{x^2+y^2}\, dy\, dx = \int_0^{\pi/2}\int_0^{2\cos\theta} \frac{\cancel{r}\cos\theta}{\cancel{r^2}}\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2}\int_0^{2\cos\theta} \cos\theta\, dr\, d\theta$$

$$= \int_0^{\pi/2} \cos\theta\, [r]_0^{2\cos\theta}\, d\theta$$

$$= \int_0^{\pi/2} \cos\theta\, [2\cos\theta]\, d\theta$$

$$= 2\int_0^{\pi/2} \cos^2\theta\, d\theta = \cancel{2}\int_0^{\pi/2} \left(\frac{1+\cos 2\theta}{\cancel{2}}\right) d\theta$$

$$= \int_0^{\pi/2} (1+\cos 2\theta)\, d\theta$$

$$= \int_0^{\pi/2} (1+\cos 2\theta)\, d\theta$$

$$= \left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= \left[\pi/2 + \frac{\sin(\cancel{2}\,\pi/\cancel{2})}{2}\right] - \left[0 + \frac{\sin(0)}{2}\right]$$

$$= \left[\pi/2 + \frac{\sin\pi}{2}\right] - [0]$$

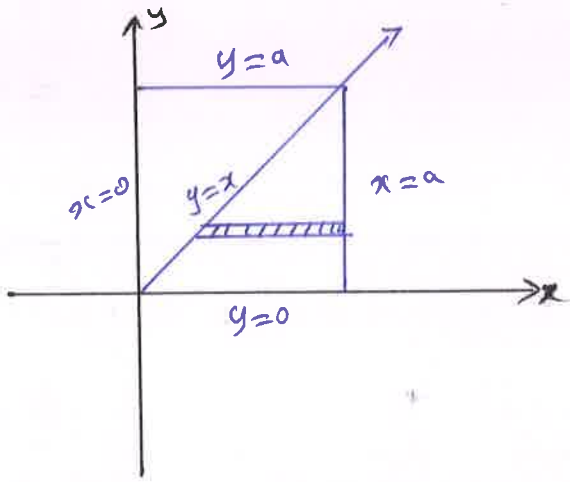
$$= [\pi/2 + 0]$$

$$\boxed{I = \pi/2}$$

Example - 6 Express $\int_0^a \int_y^a \frac{x^2}{(x^2+y^2)^{3/2}}\, dx\, dy$ in Polar Coordinates and then Evaluate.

Soln :-

Given that $\int_0^a \int_y^a \frac{x^2}{(x^2+y^2)^{3/2}}\, dx\, dy$



Given Limit :-

$y=0$ to $y=a$

$x=y$ to $x=a$

To change into Polar Coordinates

Let $x = r\cos\theta$; $y = r\sin\theta$

$r^2 = x^2+y^2$; $dx\,dy = r\,dr\,d\theta$

Here, $\boxed{r=0}$; $x = a \Rightarrow r\cos\theta = a$

$r = a/\cos\theta$

$\boxed{r = a\sec\theta}$

Limits

$\theta = 0$ to $\theta = \pi/4$

$r = 0$ to $r = a\sec\theta$

$$\int_0^a \int_y^a \frac{x^2}{(x^2+y^2)^{3/2}}\,dx\,dy = \int_0^{\pi/4} \int_0^{a\sec\theta} \frac{r^2\cos^2\theta}{(r^2)^{3/2}}\, r\,dr\,d\theta$$

$$= \int_0^{\pi/4} \int_0^{a\sec\theta} \frac{r^2\cos^2\theta}{r^3}\, r\,dr\,d\theta$$

$$= \int_0^{\pi/4} \int_0^{a\sec\theta} \cos^2\theta\, dr\,d\theta$$

$$= \int_0^{\pi/4} \cos^2\theta \left[r\right]_0^{a\sec\theta} d\theta$$

$$= \int_0^{\pi/4} \cos^2\theta \left[a\sec\theta\right] d\theta$$

$$= a\int_0^{\pi/4} \cos^2\theta \sec\theta\, d\theta$$

$$= a\int_0^{\pi/2} \cos^2\theta \frac{1}{\cos\theta}\, d\theta \qquad \left\{ \text{W.k.T } \sec\theta = \frac{1}{\cos\theta} \right.$$

$$= a\int_0^{\pi/2} \cos\theta\, d\theta$$

$$= a\left[\sin\theta\right]_0^{\pi/2}$$

$$= a\left[\sin\pi/2 - \sin(0)\right]$$

$$= a[1-0]$$

$$\boxed{I = a}$$

chapter-5.5 [Triple integration]

Example-(1) Evaluate $\int_0^2\int_1^3\int_1^2 xy^2z\,dz\,dy\,dx$

Soln

Given that $\int_0^2\int_1^3\int_1^2 xy^2z\,dz\,dy\,dx$

$$= \int_0^2\int_1^3\left[\frac{xy^2z^2}{2}\right]_1^2 dy\,dx$$

$$= \int_0^2\int_1^3\left[\frac{xy^2(2)^2}{2} - \frac{xy^2(1)^2}{2}\right] dy\,dx$$

$$= \int_0^2\int_1^3\left(\frac{4xy^2}{2} - \frac{xy^2}{2}\right) dy\,dx$$

$$= \frac{1}{2}\int_0^2\int_1^3 (4xy^2 - xy^2)\,dy\,dx$$

$$= \frac{1}{2}\int_0^2\left[\frac{4xy^3}{3} - \frac{xy^3}{3}\right]_1^3 dx$$

$$= \frac{1}{2\times3}\int_0^2\left[4xy^3 - xy^3\right]_1^3 dx$$

$$= \frac{1}{6}\int_0^2 [4x(3)^3 - x(3)^3] - [4x(1)^3 - x(1)^3]\,dx$$

$$= \frac{1}{6}\int_0^2 [4x(27) - 27x] - [4x - x]\,dx$$

$$= \frac{1}{6}\int_0^2 [108x - 27x] - (3x)\,dx$$

$$= \frac{1}{6}\int_0^2 78x\,dx$$

$$= \frac{1}{6}\int_0^2 78x\,dx = \frac{1}{6}\left[\frac{\cancel{78}^{39}x^2}{\cancel{2}}\right]_0^2$$

$$= \frac{1}{6}\left[39x^2\right]_0^2 = \frac{1}{6}\left[39(2)^2 - 0\right]$$

$$= \frac{1}{6}\left[39(4)\right] = \frac{1}{6}\left[156\right]$$

$$\boxed{I = 26}$$

Example-② Evaluate $\int_0^a\int_0^b\int_0^c (x+y+z)\,dz\,dy\,dx$

Soln.

$$\int_0^a\int_0^b\int_0^c (x+y+z)\,dz\,dy\,dx$$

$$= \int_0^a\int_0^b\left[xz + yz + \frac{z^2}{2}\right]_0^c dy\,dx$$

$$= \int_0^a\int_0^b\left[cx + cy + \frac{c^2}{2}\right] dy\,dx$$

$$= \int_0^a\left[cxy + \frac{cy^2}{2} + \frac{c^2}{2}y\right]_0^b dx$$

$$= \int_0^a\left[cbx + \frac{cb^2}{2} + c^2b\right] dx$$

$$= \left[\frac{cbx^2}{2} + \frac{b^2cx}{2} + c^2bx\right]_0^a$$

$$= \left[\frac{cba^2}{2} + \frac{b^2ca}{2} + \frac{c^2ba}{2}\right]$$

$$= \frac{1}{2}\left[a^2bc + ab^2c + abc^2\right]$$

Example-5 Evaluate $\int_0^{2a}\int_0^{x}\int_y^{x} xyz\, dx\, dy\, dz$

Soln:
$$\int_0^{2a}\int_0^{x}\int_y^{x} xyz\, dx\,dy\,dz = \int_0^{2a}\int_0^{x} xyz\, dz\,dy\,dx$$

$$= \int_0^{2a}\int_0^{x}\left[\frac{xyz^2}{2}\right]_y^x dy\,dx = \frac{1}{2}\int_0^{2a}\int_0^{x}\left[x\,yz^2\right]_y^x dy\,dx$$

$$= \frac{1}{2}\int_0^{2a}\int_0^{x}\left[xy\,x^2 - xy\,y^2\right] dy\,dx$$

$$= \frac{1}{2}\int_0^{2a}\int_0^{x}\left[x^3y - xy^3\right] dy\,dx$$

$$= \frac{1}{2}\int_0^{2a}\left[\frac{x^3y^2}{2} - \frac{xy^4}{4}\right]_0^x dx$$

$$= \frac{1}{2}\int_0^{2a}\left[\frac{x^3x^2}{2} - \frac{x\,x^4}{4}\right] dx$$

$$= \frac{1}{2}\int_0^{2a}\left[\frac{x^5}{2} - \frac{x^5}{4}\right] dx$$

$$= \frac{1}{2}\left[\frac{x^6}{12} - \frac{x^6}{24}\right]_0^{2a} = \frac{1}{2}\left[\frac{(2a)^6}{12} - \frac{(2a)^6}{24}\right]$$

$$= \frac{1}{2}\left[\frac{64a^6}{12} - \frac{64a^6}{24}\right] = \frac{a^6}{2}\left[\frac{64}{12} - \frac{64}{24}\right]$$

$$= \frac{a^6}{2}\left[\frac{16}{3} - \frac{8}{3}\right] = \frac{a^6}{\cancel{2}}\left[\frac{\cancel{8}^{\,4}}{3}\right] = a^6\left(\frac{4}{3}\right)$$

$$I = \frac{4}{3}a^6$$

Example-(4)

Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{a^2-x^2-y^2-z^2}}$

Soln

Given that $\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{a^2-x^2-y^2-z^2}}$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\sin^{-1}\frac{z}{\sqrt{a^2-x^2-y^2}}\right]_0^{\sqrt{a^2-x^2-y^2}} dy\,dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\sin^{-1}\left(\frac{\sqrt{a^2-x^2-y^2}}{\sqrt{a^2-x^2-y^2}}\right) - \sin^{-1}\left(\frac{0}{\sqrt{a^2-x^2-y^2}}\right)\right] dy\,dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\sin^{-1}(1) - \sin^{-1}(0)\right] dy\,dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\pi/2 - 0\right] dy\,dx = \int_0^a \int_0^{\sqrt{a^2-x^2}} \pi/2 \; dy\,dx$$

$$= \pi/2 \int_0^a \int_0^{\sqrt{a^2-x^2}} dy\,dx = \pi/2 \int_0^a \left[y\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \pi/2 \int_0^a \sqrt{a^2-x^2}\,dx$$

$$= \pi/2 \left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}(x/a)\right]_0^a$$

$$= \pi/2 \left\{\left[\frac{a}{2}\sqrt{a^2-a^2} + \frac{a^2}{2}\sin^{-1}(a/a)\right] - \left[0 + \frac{a^2}{2}\sin^{-1}(0)\right]\right\}$$

$$= \pi/2 \left\{\left[0 + \frac{a^2}{2}\sin^{-1}(1)\right] - 0\right\}$$

$$= \pi/2 \left[\frac{a^2}{2}\,\pi/2\right] = \frac{\pi a^2}{8}$$

Example-5 Evaluate $\int_0^1 \int_0^{1-x} \int_0^{x+y} e^z \, dx\, dy\, dz$

Soln:

Given that $\int_0^1 \int_0^{1-x} \int_0^{x+y} e^z \, dx\, dy\, dz$

$$= \int_0^1 \int_0^{1-x} \int_0^{x+y} e^z \, dz\, dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \left[e^z\right]_0^{x+y} dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \left[e^{x+y} - e^0\right] dy\, dx = \int_0^1 \int_0^{1-x} \left[e^{x+y} - 1\right] dy\, dx$$

$$= \int_0^1 \int_0^{1-x} (e^x e^y - 1)\, dy\, dx$$

$$= \int_0^1 \left[e^x e^y - y\right]_0^{1-x} dx = \int_0^1 \left[e^x e^{1-x} - (1-x)\right] - \left[e^x e^0 - 0\right] dx$$

$$= \int_0^1 \left[e^{x+1-x} - (1-x)\right] - \left[e^x (1)\right] dx$$

$$= \int_0^1 \left[e^1 - 1 + x - e^x\right] dx = \int_0^1 \left[e - 1 + x - e^x\right] dx$$

$$= \left[ex - x + \frac{x^2}{2} - e^x\right]_0^1$$

$$= \left[e(1) - 1 + \tfrac{1}{2} - e^1\right] - \left[e(0) - 0 + \tfrac{0}{2} - e^0\right]$$

$$= \left[\cancel{e} - 1 + \tfrac{1}{2} - \cancel{e}\right] - \left[-e^0\right]$$

$$= \cancel{-1} + \tfrac{1}{2} + \cancel{1}$$

$$\boxed{I = \tfrac{1}{2}}$$

Example-6 Evaluate $\iiint \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}}$ over the first octant of the sphere $x^2+y^2+z^2=1$.

Soln

Given that first octant of the sphere $x^2+y^2+z^2=1 \longrightarrow (1)$

Put $z=y=0$ in Eqn (1) $\quad 0+x^2=1 \Rightarrow x^2=1$

to $\boxed{x=\pm 1}$, $\boxed{x=1}$

Put $z=0$ in Eqn (1) $\quad x^2+y^2=1 \Rightarrow y^2=1-x^2$

$y=\pm\sqrt{1-x^2}$

to $\boxed{y=\sqrt{1-x^2}}$

Eqn (1) $\Rightarrow x^2+y^2+z^2=1$

$z^2=1-x^2-y^2$

$z=\pm\sqrt{1-x^2-y^2}$

to $\boxed{z=\sqrt{1-x^2-y^2}}$

Limits are:

$x=0$ to $x=1$

$y=0$ to $y=\sqrt{1-x^2}$

$z=0$ to $z=\sqrt{1-x^2-y^2}$

$$\iiint \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}} = \int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2-z^2}} \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}}$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\sin^{-1}\left(\frac{z}{\sqrt{1-x^2-y^2-z^2}}\right)\right]_0^{\sqrt{1-x^2-y^2-z^2}} dy\,dx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\sin^{-1}\left(\frac{\sqrt{1-x^2-y^2-z^2}}{\sqrt{1-x^2-y^2-z^2}}\right) - \sin^{-1}(0)\right] dy\,dx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\sin^{-1}(1) - \sin^{-1}(0)\right] dy\,dx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} [\pi/2 - 0]\, dy\,dx = \int_0^1 \int_0^{\sqrt{1-x^2}} \pi/2 \, dy\,dx$$

$$= \pi/2 \int_0^1 \int_0^{\sqrt{1-x^2}} dy\,dx = \pi/2 \int_0^1 [y]_0^{\sqrt{1-x^2}} dx$$

$$= \pi/2 \int_0^1 \left(\sqrt{1-x^2}\right) dx = \pi/2 \left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x\right]_0^1$$

$$= \pi/2 \left\{ \left[\tfrac{1}{2}\sqrt{1-1} + \tfrac{1}{2}\sin^{-1}(1)\right] - \left[0 + \tfrac{1}{2}\sin^{-1}(0)\right] \right\}$$

$$= \pi/2 \left\{ [0 + \tfrac{1}{2}(\pi/2)] - [0] \right\}$$

$$= \pi/2 \, [\pi/4]$$

$$I = \frac{\pi^2}{8} \;//$$

Example-(7) Evaluate $\int_0^a\int_0^b\int_0^c xyz\, dz\, dy\, dx$

Soln:

Given that $\int_0^a\int_0^b\int_0^c xyz\, dz\, dy\, dx$

$$= \int_0^a\int_0^b \left[\frac{xyz^2}{2}\right]_0^c dy\, dx$$

$$= \int_0^a\int_0^b \left[\frac{xyc^2}{2}\right] dy\, dx$$

$$= \int_0^a \left[\frac{c^2xy^2}{4}\right]_0^b dx$$

$$= \int_0^a \left[\frac{c^2xb^2}{4}\right] dx$$

$$= \frac{b^2c^2}{4}\int_0^a x\, dx = \frac{b^2c^2}{4}\left[\frac{x^2}{2}\right]_0^a$$

$$= \frac{b^2c^2}{4}\left[a^2/2\right]$$

$$= \frac{a^2b^2c^2}{8}$$

$$I = \frac{(abc)^2}{8}$$

Chapter - ~~5~~ 6 [Volume as a Trible Integral]

Example - ①

Find the volume of the sphere $x^2+y^2+z^2=a^2$ using triple integration.

Soln

Volume $(V) = 8 \times$ volume of an octant.

Limits

$x=0$ to $x=a$

$y=0$ to $y=\sqrt{a^2-x^2}$

$z=0$ to $z=\sqrt{a^2-x^2-y^2}$

$$V = 8\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx$$

$$= 8\int_0^a\int_0^{\sqrt{a^2-x^2}} \left[z\right]_0^{\sqrt{a^2-x^2-y^2}} dy\,dx$$

$$= 8\int_0^a\int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\, dy\,dx$$

$\left\{ \text{W.K.T} \quad \int \sqrt{a^2-x^2}\,dx = \frac{a^2}{2}\sin^{-1}x/a + x/2\sqrt{a^2-x^2} \right.$

$$= 8\int_0^a\int_0^{\sqrt{a^2-x^2}} \sqrt{(a^2-x^2)-y^2}\, dy\,dx$$

$$= 8\int_0^a \left[\frac{a^2-x^2}{2}\sin^{-1}\left(\frac{y}{\sqrt{a^2-x^2}}\right) + y/2\sqrt{(a^2-x^2)-y^2}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= 8\int_0^a \left\{\left[\frac{a^2-x^2}{2}\sin^{-1}\frac{\sqrt{a^2-x^2}}{\sqrt{a^2-x^2}} + \frac{\sqrt{a^2-x^2}}{2}\sqrt{(a^2-x^2)-(a^2-x^2)}\right]\right.$$

$$\left. - \left[\frac{a^2-x^2}{2}\sin^{-1}(0) + (0)\sqrt{a^2-x^2-0}\right]\right\} dx$$

$$= 8\int_0^a \left[\frac{a^2-x^2}{2} \sin^{-1}(1) - (0)\right] - [0]\, dx$$

$$= 8\int_0^a \left(\frac{a^2-x^2}{2}\right)(\pi/2)\, dx$$

$$= \frac{2\cancel{8}\pi}{\cancel{4}}\int_0^a (a^2-x^2)\, dx$$

$$= 2\pi\left[a^2x - \frac{x^3}{3}\right]_0^a = 2\pi\left[a^2(a) - a^3/3\right] - [0]$$

$$= 2\pi\left[a^3 - a^3/3\right] = 2\pi\left[\frac{3a^3-a^3}{3}\right]$$

$$= 2\pi\left[\frac{2a^3}{3}\right] = \frac{4\pi a^3}{3}$$

$$\boxed{V = \frac{4\pi a^3}{3}}$$

Example - ② Evaluate $\iiint xyz\, dx\,dy\,dz$ over the first octant of $x^2+y^2+z^2=a^2$.

Soln:

Given that first octant of the sphere $x^2+y^2+z^2=a^2$.

Limits:-

$x=0$ to $x=a$

$y=0$ to $y=\sqrt{a^2-x^2}$

$z=0$ to $z=\sqrt{a^2-x^2-y^2}$

$$\iiint xyz\,dx\,dy\,dz = \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} xyz \; dz\,dy\,dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\frac{xyz^2}{2}\right]_0^{\sqrt{a^2-x^2-y^2}} dy\,dx$$

$$= \frac{1}{2}\int_0^a \int_0^{\sqrt{a^2-x^2}} \left[xyz^2\right]_0^{\sqrt{a^2-x^2-y^2}} dy\,dx$$

$$= \frac{1}{2}\int_0^a \int_0^{\sqrt{a^2-x^2}} xy\left(\sqrt{a^2-x^2-y^2}\right)^2 dy\,dx$$

$$= \frac{1}{2}\int_0^a \int_0^{\sqrt{a^2-x^2}} xy(a^2-x^2-y^2)\,dy\,dx$$

$$= \frac{1}{2}\int_0^a \int_0^{\sqrt{a^2-x^2}} (a^2xy - x^3y - xy^3)\,dy\,dx$$

$$= \frac{1}{2}\int_0^a \left[\frac{a^2xy^2}{2} - \frac{x^3y^2}{2} - \frac{xy^4}{4}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2}\int_0^a \left[\frac{a^2x\left(\sqrt{a^2-x^2}\right)^2}{2} - \frac{x^3\left(\sqrt{a^2-x^2}\right)^2}{2} - \frac{x\left(\sqrt{a^2-x^2}\right)^4}{4}\right] dx$$

$$= \frac{1}{2}\int_0^a \left[\frac{a^2x(a^2-x^2)}{2} - \frac{x^3(a^2-x^2)}{2} - \frac{x(a^2-x^2)^2}{4}\right] dx$$

$$= \frac{1}{2}\int_0^a \left[\frac{a^4x - a^2x^3}{2} - \left(\frac{a^2x^3 - x^5}{2}\right) - \frac{x\left[a^4 + x^4 - 2a^2x^2\right)}{4}\right] dx$$

$$= \frac{1}{2}\int_0^a \left[\frac{a^4x - a^2x^3 - a^2x^3 + x^5}{2} - \frac{(a^4x + x^5 - 2a^2x^3)}{4}\right] dx$$

$$= \frac{1}{2}\int_0^a \left[\frac{a^4x - 2a^2x^3 + x^5}{2} - \frac{(a^4x + x^5 - 2a^2x^3)}{4}\right] dx$$

$$= \frac{1}{2}\int_0^a \frac{2(a^4x - 2a^2x^3 + x^5) - (a^4x + x^5 - 2a^2x^3)}{4}\, dx$$

$$= \frac{1}{8}\int_0^a (2a^4x - 4a^2x^3 + 2x^5 - a^4x - x^5 + 2a^2x^3)\, dx$$

$$= \frac{1}{8}\int_0^a (a^4x - 2a^2x^3 + x^5)\, dx$$

$$= \frac{1}{8}\left[\frac{a^4x^2}{2} - \frac{2a^2x^4}{4} + \frac{x^6}{6}\right]_0^a$$

$$= \frac{1}{8}\left[\frac{a^4(a^2)}{2} - \frac{\cancel{2}a^2(a^4)}{\cancel{4}2} + \frac{a^6}{6}\right]$$

$$= \frac{1}{8}\left[\cancel{\frac{a^6}{2}} - \cancel{\frac{a^6}{2}} + \frac{a^6}{6}\right]$$

$$= \frac{1}{8}\left[\frac{a^2}{6}\right]$$

$$\boxed{V = \frac{a^2}{48}}$$

Example - 3 Evaluate $\iiint_V dx\,dy\,dz$, where V is the finite region of space formed by the planes $x=0$; $y=0$; $z=0$ and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

Soln. Given that $x=0$; $y=0$; $z=0$ and

$$x/a + y/b + z/c = 1.$$

Put, $\boxed{y=0}$; $\boxed{z=0}$; $\Rightarrow \frac{x}{a}+0=1 \Rightarrow x/a=1 \Rightarrow \boxed{x=a}$

$\boxed{z=0} \Rightarrow x/a + y/b + 0 = 1 \Rightarrow x/a + y/b = 1$

$y/b = 1 - x/a$

$\boxed{y = b(1-x/a)}$

$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \Rightarrow z/c = 1 - x/a - y/b$

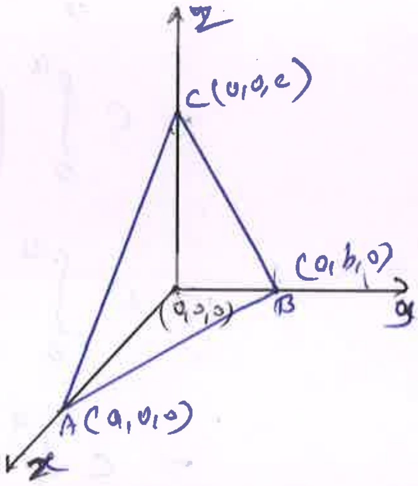
$\boxed{z = c[1-x/a-y/b]}$

Limits are :-

$x=0$ to $x=a$

$y=0$ to $y=b(1-x/a)$

$z=0$ to $z=c[1-x/a-y/b)$



$$\iiint dx\,dy\,dz = \int_0^a \int_0^{b(1-x/a)} \int_0^{c(1-x/a-y/b)} dz\,dy\,dx$$

$$= \int_0^a \int_0^{b(1-x/a)} [z]_0^{c[1-x/a-y/b)} dy\,dx$$

$$= \int_0^a \int_0^{b(1-x/a)} c(1-x/a-y/b)\cdot dy\,dx$$

$$= c\int_0^a \int_0^{b(1-x/a)} (1-x/a-y/b)\,dy\,dx$$

$$= c\int_0^a \left[y - \frac{xy}{a} - \frac{y^2}{2b}\right]_0^{b(1-x/a)} dx$$

$$= c\int_0^a \left[b(1-x/a) - \frac{x}{a}b(1-x/a) - \frac{[b(1-x/a)]^2}{2b}\right] dx$$

$$= c\int_0^a \left[b(1-x/a) - \frac{bx(1-x/a)}{a} - \frac{b^2(1-x/a)^2}{2b}\right] dx$$

$$= c\int_0^a \left[b\left(\frac{a-x}{a}\right) - \frac{bx\left(\frac{a-x}{a}\right)}{a} - \frac{b^2\left(\frac{a-x}{a}\right)^2}{2b}\right] dx$$

$$= c\int_0^a \left[\frac{\cancel{a}b}{\cancel{a}} - \frac{bx}{a} - \frac{\cancel{a}bx}{a\cancel{2}} + \frac{bx^2}{a^2} - \frac{b^{\cancel{2}}(a-x)^2}{2a^2\cancel{b}}\right] dx$$

$$= c\int_0^a \left[b - \frac{bx}{a} - \frac{bx}{a} + \frac{bx^2}{a^2} - \frac{b(a-x)^2}{2a^2}\right] dx$$

$$= c\int_0^a \left[b - \frac{2bx}{a} + \frac{bx^2}{a^2} - \frac{b(a^2+x^2-2ax)}{2a^2}\right] dx$$

$$= c\int_0^a \left[b - \frac{2bx}{a} + \frac{bx^2}{a^2} - \frac{a^{\cancel{2}}b}{2\cancel{a^2}} - \frac{bx^2}{2a^2} + \frac{\cancel{2a}bx}{\cancel{2a^2}}\right] dx$$

$$= c\int_0^a \left[b - \frac{2bx}{a} + \frac{bx^2}{a^2} - \frac{ab}{2} - \frac{bx^2}{2a^2} + \frac{bx}{a}\right] dx$$

$$= c\left[b(x) - \frac{2b}{a}\left(\frac{x^2}{2}\right) + \frac{b}{a^2}\left(\frac{x^3}{3}\right) - \frac{ab}{2}(x) - \frac{b}{2a^2}\left(\frac{x^3}{3}\right) + \frac{b}{a}\left(\frac{x^2}{2}\right)\right]_0^a$$

$$= c\left[ba - \frac{\cancel{2}b}{\cancel{a}}\left(\frac{a^{\cancel{2}}}{\cancel{2}}\right) + \frac{b}{\cancel{a^2}}\left(\frac{a^{\cancel{3}}}{3}\right) - \frac{ab}{2}(a) - \frac{b}{2\cancel{a^2}}\left(\frac{a^{\cancel{3}}}{3}\right) + \frac{b}{\cancel{a}}\left(\frac{a^{\cancel{2}}}{2}\right)\right]$$

$$= c\left[\cancel{ab} - \cancel{ab} + \frac{ab}{3} - \cancel{\frac{a^2b}{2}} - \frac{ab}{6} + \cancel{\frac{a^2b}{2}}\right]$$

$$= c\left[\frac{ab}{3} - \frac{ab}{6}\right] = c\left[\frac{2ab-ab}{6}\right]$$

$$= c\left[\frac{ab}{6}\right]$$

$$\boxed{V = \frac{abc}{6}}$$

Example-(4) Evaluate $\iiint_V dx\,dy\,dz$, where V is the finite region of space (tetrahedron) bounded by the planes $x=0$; $y=0$; $z=0$ and $2x+3y+4z=12$.

Soln.

Given curves are $x=0$; $y=0$; $z=0$ and

$$2x+3y+4z=12.$$

The intercept form of the plane $2x+3y+4z=12$

$$\div \text{ by } 12,\quad \frac{2x}{12}+\frac{3y}{12}+\frac{4z}{12}=\frac{12}{12}$$

$$\frac{x}{6}+\frac{y}{4}+\frac{z}{3}=1 \longrightarrow (1)$$

Put $\boxed{x=0}$, $\boxed{y=0}$ in Eqn (1)

$$(1) \Rightarrow 0+0+\frac{z}{3}=1 \Rightarrow \boxed{z=3}$$

Put $\boxed{x=0}$ in Eqn (1)

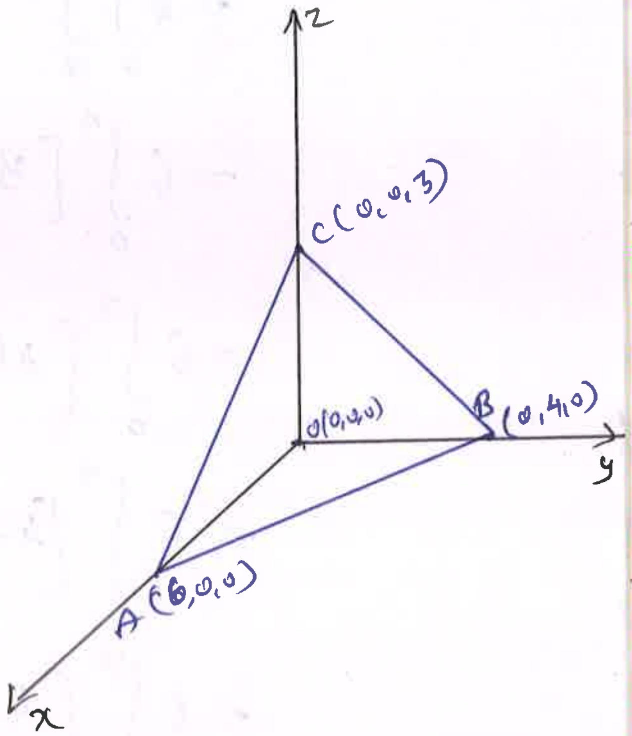
$$(1) \Rightarrow \frac{y}{4}+\frac{z}{3}=1 \Rightarrow \frac{y}{4}=1-z/3$$

$$\boxed{y=4(1-z/3)}$$

$$(1) \Rightarrow \frac{x}{6}+\frac{y}{4}+\frac{z}{3}=1$$

$$\Rightarrow \frac{x}{6}=1-y/4-z/3$$

$$\boxed{x=6(1-y/4-z/3)}$$



Limits:
$z = 0$ to $z = 3$
$y = 0$ to $y = 4(1 - z/3)$
$z = 0$ to $z = 6(1 - y/4 - z/3)$

$$\iiint_V dx\,dy\,dz = \int_0^3 \int_0^{4(1-z/3)} \int_0^{6(1-y/4-z/3)} dx\,dy\,dz$$

$$= \int_0^3 \int_0^{4(1-z/3)} \left[x\right]_0^{6(1-y/4-z/3)} dy\,dz$$

$$= \int_0^3 \int_0^{4(1-z/3)} 6(1 - y/4 - z/3)\,dy\,dz$$

$$= 6 \int_0^3 \int_0^{4(1-z/3)} (1 - y/4 - z/3)\,dy\,dz$$

$$= 6 \int_0^3 \left[y - \frac{y^2}{8} - \frac{zy}{3}\right]_0^{4(1-z/3)} dz$$

$$= 6 \int_0^3 \left[4(1-z/3) - \frac{(4(1-z/3))^2}{8} - \frac{z\,4(1-z/3)}{3}\right] dz$$

$$= 6 \int_0^3 \left[4 - \frac{4z}{3} - \frac{2\cancel{16}(1-z/3)^2}{\cancel{8}} - \frac{4z(1-z/3)}{3}\right] dz$$

$$= 6 \int_0^3 \left[4 - \frac{4z}{3} - 2\left(\frac{3-z}{3}\right)^2 - \frac{4z\left(\frac{3-z}{3}\right)}{3}\right] dz$$

$$= 6 \int_0^3 \left[4 - \frac{4z}{3} - \frac{2(3-z)^2}{9} - \frac{4z(3-z)}{9}\right] dz$$

$$= 6\int_0^3 \left[4 - \frac{4z}{3} - \frac{2(9+z^2-6z)}{9} - \frac{12z + 4z^2}{9}\right] dz$$

$$= 6\int_0^3 \left[4 - \frac{4z}{3} - \frac{18 - 2z^2 + 12z}{9} - \frac{\cancel{12}^{4}z}{\cancel{9}_{3}} + \frac{4z^2}{9}\right] dz$$

$$= 6\int_0^3 \left[4 - \frac{4z}{3} - \frac{\cancel{18}^{2}}{\cancel{9}} - \frac{2z^2}{9} + \frac{\cancel{12}^{4}z}{\cancel{9}_{3}} - \frac{\cancel{1}4z}{3} + \frac{4z^2}{9}\right] dz$$

$$= 6\int_0^3 \left[4 - \frac{4z}{3} - 2 - \frac{2z^2}{9} + \cancel{\frac{4z}{3}} - \cancel{\frac{4z}{3}} + \frac{4z^2}{9}\right] dz$$

$$= 6\int_0^3 \left[2 - \frac{4z}{3} + \frac{2z^2}{9}\right] dz$$

$$= 6\left[2z - \frac{4z^2}{6} + \frac{2z^3}{27}\right]_0^3$$

$$= 6\left[2(3) - \frac{4(3)^2}{6} + \frac{2(3)^3}{27}\right]$$

$$= 6\left[6 - \frac{\cancel{4}^{2}(9)}{\cancel{6}_{3}} + \frac{2\cancel{(27)}}{\cancel{27}}\right] = 6\left[6 - \frac{2\cancel{(9)}^{3}}{\cancel{3}} + 2\right]$$

$$= 6[\cancel{6} - \cancel{6} + 2]$$

$$= 6(2)$$

$$\boxed{V = 12}$$

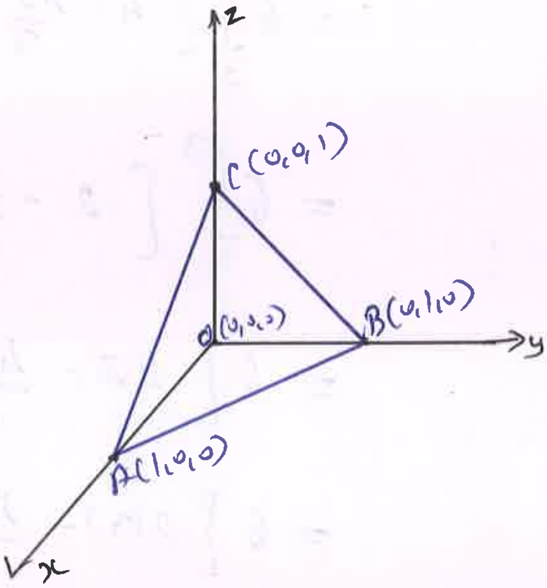
Example (5) Using the triple integration, find the volume of the tetrahedron with vertices $(0,0,0), (1,0,0), (0,1,0), (0,0,1)$.

Sol'n:

Given that $(0,0,0), (1,0,0), (0,1,0), (0,0,1)$ are points through the planes as $x+y+z=1 \longrightarrow (1)$

Put $\boxed{x=0}$; $\boxed{y=0}$ in Eqn (1)

$(1) \Rightarrow 0+0+z=1 \Rightarrow \boxed{z=1}$



Put $\boxed{x=0}$ in Eqn (1)

$(1) \Rightarrow 0+y+z=1$

$\Rightarrow \boxed{y=1-z}$

$(1) \Rightarrow x+y+z=1$

$\boxed{x=1-y-z}$

Limits:

$z=0$ to $z=1$

$y=0$ to $y=1-z$

$x=0$ to $x=1-y-z$

$$\iiint_V dx\,dy\,dz = \int_0^1 \int_0^{1-z} \int_0^{1-y-z} dx\,dy\,dz$$

$$= \int_0^1 \int_0^{1-z} [x]_0^{1-y-z}\, dy\,dz$$

$$= \int_0^1 \int_0^{1-z} [1-y-z]\, dy\,dz$$

$$= \int_0^1 \left[y - \frac{y^2}{2} - zy \right]_0^{1-z} dz$$

$$= \int_0^1 \left[y - \frac{y^2}{2} - zy \right]_0^{1-z} dz$$

$$= \int_0^1 \left[(1-z) - \frac{(1-z)^2}{2} - z(1-z) \right] dz$$

$$= \int_0^1 \left[1-z - \frac{(1+z^2-2z)}{2} - z + z^2 \right] dz$$

$$= \int_0^1 \left[1 - z - \frac{1}{2} - \frac{z^2}{2} + \frac{\cancel{2}z}{\cancel{2}} - z + z^2 \right] dz$$

$$= \int_0^1 \left[1 - z - \frac{1}{2} - \frac{z^2}{2} + \cancel{z} - \cancel{z} + z^2 \right] dz$$

$$= \int_0^1 \left[\frac{1}{2} - z - \frac{z^2}{2} + z^2 \right] dz$$

$$= \left[\frac{1}{2} z - \frac{z^2}{2} - \frac{z^3}{6} + \frac{z^3}{3} \right]_0^1$$

$$= \left[\frac{1}{2}(1) - \frac{1}{2} - \frac{1}{6} + \frac{1}{3} \right]$$

$$= \left[\cancel{\frac{1}{2}} - \cancel{\frac{1}{2}} - \frac{1}{6} + \frac{1}{3} \right] = \left[-\frac{1}{6} + \frac{1}{3} \right]$$

$$= \left[\frac{-1+2}{6} \right] = \frac{1}{6}$$

$$\boxed{V = \frac{1}{6}}$$

Cylindrical and Rectangular coordinates are related by $x = r\cos\theta$; $y = r\sin\theta$; $z = z$.

$$\iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint f(r\cos\theta, r\sin\theta, z)\, r\,dr\,d\theta\,dz.$$

Example-⑥

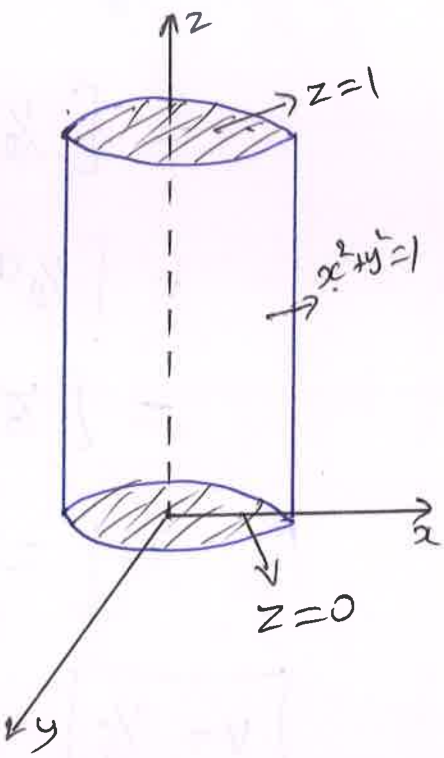
By transforming into cylindrical coordinates, Evaluate the integral $\iiint (x^2+y^2+z^2)\,dx\,dy\,dz$ taken over the region space defined by $x^2+y^2 \le 1$ and $0 \le z \le 1$.

Soln:

Here the region of space is enclosed by the cylinder $x^2+y^2=1$ and the planes $z=0$ and $z=1$.

The radius of cylinder is 1.

Putting $x = r\cos\theta$; $y = r\sin\theta$; $z=z$,



(i) $x^2+y^2=1 \Rightarrow (r\cos\theta)^2 + (r\sin\theta)^2 = 1$

$$r^2\cos^2\theta + r^2\sin^2\theta = 1$$

$$r^2(\cos^2\theta + \sin^2\theta) = 1$$

$$r^2 = 1$$

$$\boxed{r = \pm 1}$$

(ii) $x^2+y^2+z^2 \Rightarrow r^2\cos^2\theta + r^2\sin^2\theta + z^2$

$$\Rightarrow r^2(\cos^2\theta + \sin^2\theta) + z^2$$

$$\Rightarrow r^2(1) + z^2$$

$$\boxed{\Rightarrow r^2 + z^2.}$$

(iii) $\boxed{dx\,dy\,dz = r\,dr\,d\theta\,dz}$

Limits are :-

$r=0$ to $r=1$

$\theta=0$ to $\theta=2\pi$

$z=0$ to $z=1$

$$\iiint_V (x^2+y^2+z^2)\,dx\,dy\,dz = \int_0^1\int_0^{2\pi}\int_0^1 (r^2+z^2)\, r\,dr\,d\theta\,dz$$

$$= \int_0^1\int_0^{2\pi}\int_0^1 (r^3+z^2r)\,dr\,d\theta\,dz$$

$$= \int_0^1\int_0^{2\pi}\left[\frac{r^4}{4}+z^2\frac{r^2}{2}\right]_0^1 d\theta.dz$$

$$= \int_0^1\int_0^{2\pi}\left[\frac{1}{4}+\frac{z^2(1)}{2}\right] d\theta\,dz$$

$$= \int_0^1\int_0^{2\pi}\left[\frac{1}{4}+\frac{z^2}{2}\right] d\theta\,dz$$

$$= \int_0^1\left[\tfrac{1}{4}\theta+\frac{z^2}{2}\theta\right]_0^{2\pi} dz$$

$$= \int_0^1\left[\tfrac{1}{\cancel{4}_2}(\cancel{2}\pi)+\frac{z^2}{\cancel{2}}(\cancel{2}\pi)\right] dz$$

$$= \int_0^1\left[\pi/2+z^2\pi\right] dz$$

$$= \left[\pi/2\,z+\pi\frac{z^3}{3}\right]_0^1$$

$$= \left[\pi/2(1)+\pi(1/3)\right] = \pi/2+\pi/3$$

$$= \frac{3\pi+2\pi}{6}$$

$$\boxed{=\frac{5\pi}{6}}$$

Example-(7) Evaluate $\iiint\limits_V xyz \, dx\,dy\,dz$ over the positive octant of the sphere $x^2+y^2+z^2=a^2$ by using cylindrical coordinates.

Soln:
Let us transform to cylindrical coordinates
then $x = r\cos\theta$; $y = r\sin\theta$; $z = z$; and

$$dx\,dy\,dz = r\,dr\,d\theta\,dz.$$

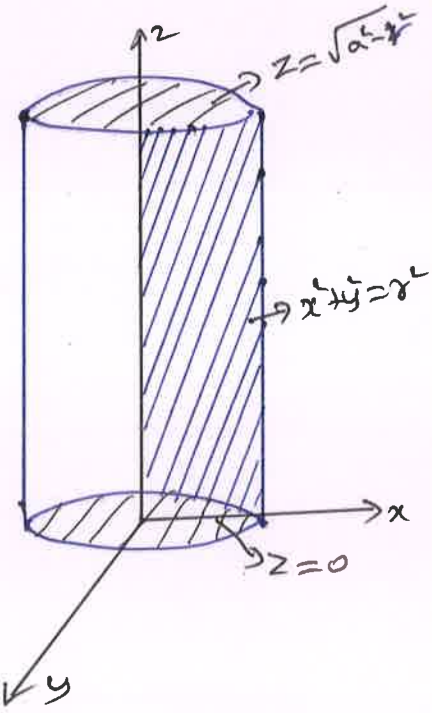
$\{$ w.k.t $x^2+y^2=r^2$

Now, $x^2+y^2+z^2=a^2$

$\Rightarrow r^2+z^2=a^2$

$\Rightarrow z^2 = a^2-r^2$

$\Rightarrow \boxed{z = \pm\sqrt{a^2-r^2}}$

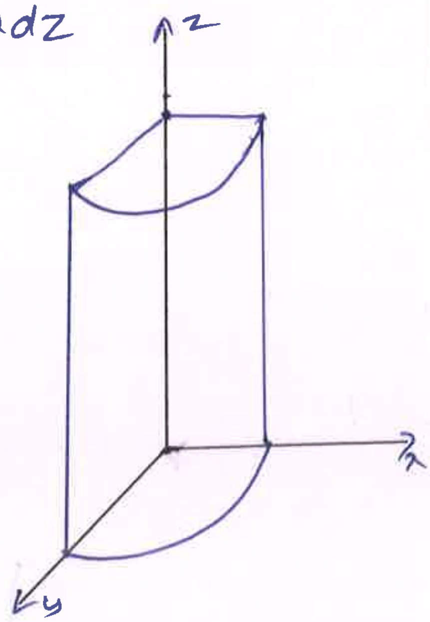


Limits are:
$z=0$ to $z=\sqrt{a^2-z^2}$
$r=0$ to $r=a$
$\theta=0$ to $\theta=\pi/2$

$$\iiint\limits_V xyz\,dx\,dy\,dz = \int_0^{\pi/2}\int_0^{a}\int_0^{\sqrt{a^2-r^2}} (r\cos\theta)(r\sin\theta)\,z\;r\,dr\,d\theta\,dz$$

$$= \int_0^{\pi/2}\int_0^{a}\int_0^{\sqrt{a^2-r^2}} z\,r^3\cos\theta\sin\theta\,dr\,d\theta\,dz$$

$$= \int_0^{\pi/2}\int_0^{a} r^3\cos\theta\sin\theta\left[\int_0^{\sqrt{a^2-r^2}} z\,dz\right] dr\,d\theta$$



$$= \int_0^{\pi/2} \int_0^a r^3 \cos\theta \sin\theta \left[\frac{z^2}{2} \right]_0^{\sqrt{a^2-r^2}} dr\, d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} \int_0^a r^3 \cos\theta \sin\theta \left[z^2 \right]_0^{\sqrt{a^2-r^2}} dr\, d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} \int_0^a r^3 \cos\theta \sin\theta \left(\sqrt{a^2-r^2}\right)^2 dr\, d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} \int_0^a r^3 \cos\theta \sin\theta \,(a^2-r^2)\, dr\, d\theta.$$

$$= \frac{1}{2} \int_0^{\pi/2} \int_0^a \cos\theta \sin\theta \,(a^2 r^3 - r^5)\, dr\, d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} \int_0^a \frac{\sin 2\theta}{2} (a^2 r^3 - r^5)\, dr\, d\theta$$

$$= \frac{1}{4} \int_0^{\pi/2} \int_0^a \sin 2\theta \,(a^2 r^3 - r^5)\, dr\, d\theta$$

$$= \frac{1}{4} \left[\int_0^{\pi/2} \sin 2\theta \, d\theta \right] \left[\int_0^a (a^2 r^3 - r^5)\, dr \right]$$

$$= \frac{1}{4} \left[-\frac{\cos 2\theta}{2} \right]_0^{\pi/2} \left[\frac{a^2 r^4}{4} - \frac{r^6}{6} \right]_0^a$$

$$= \frac{1}{4} \left[\frac{-\cos 2(\pi/2) - (-\cos 0)}{2} \right] \left[\frac{a^2 a^4}{4} - \frac{a^6}{6} \right]$$

$$= \frac{1}{4} \left[\frac{-\cos \pi + \cos(0)}{2} \right] \left[\frac{a^6}{4} - \frac{a^6}{6} \right]$$

$$= \frac{1}{4} \left[\frac{-(-1)+1}{2} \right] \left[\frac{6a^6 - 4a^6}{24} \right]$$

$$= \frac{1}{4} \left[\frac{1+1}{2} \right] \left[\frac{2a^6}{24_{12}} \right] = \frac{1}{4} \left[\frac{2}{2} \right] \left[\frac{a^6}{12} \right] = \frac{1}{4} \left(\frac{a^6}{12} \right)$$

$$\boxed{= \frac{a^6}{48}}$$

Spherical Coordinates :-

Let $x = r\sin\phi\cos\theta$; $y = r\sin\phi\sin\theta$; $z = r\cos\phi$.

then $\iiint\limits_V f(x, y, z)\, dx\, dy\, dz$

$= \iiint\limits_V f(r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi)\, r^2 \sin\phi\, dr\, d\phi\, d\theta$.

Example-⑧ Evaluate $\iiint \frac{dx\,dy\,dz}{x^2+y^2+z^2}$ throughout the volume of the sphere $x^2+y^2+z^2=a^2$ transforming into spherical coordinates.

Soln:-

In spherical polar coordinates systems, we have

$x = r\sin\phi\cos\theta$; $y = r\sin\phi\sin\theta$; $z = r\cos\phi$

$dx\,dy\,dz = r^2\sin\phi\, dr\, d\phi\, d\theta$.

Given that $x^2+y^2+z^2 = r^2$.

Limits are: $r = 0$ to $r = a$

$\theta = 0$ to $\theta = 2\pi$

$\phi = 0$ to $\phi = \pi$

$$\iiint \frac{dx\,dy\,dz}{x^2+y^2+z^2} = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \frac{\cancel{r^2}\sin\phi\, dr\, d\phi\, d\theta}{\cancel{r^2}}$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \sin\phi\, dr\, d\phi\, d\theta$$

$$= \left[\int_0^{a} dr\right]\left[\int_0^{\pi}\sin\phi\, d\phi\right]\left[\int_0^{2\pi} d\theta\right]$$

$$= [r]_0^a \,[-\cos\phi]_0^{\pi}\, [\theta]_0^{2\pi}$$

$$= [a-0]\,[-\cos\pi + \cos 0]\,[2\pi - 0]$$

$$= a\,[-(-1)+1]\,[2\pi] = a(1+1)\,2\pi = 2a(2\pi)$$

$$= 4\pi a$$

Example - (9) Find the volume of the sphere $x^2+y^2+z^2=a^2$ by transforming into spherical polar coordinates.

Soln:

In spherical polar coordinates system, we have

$$x = r\sin\phi\cos\theta \;;\; y = r\sin\phi\sin\theta \;;\; z = r\cos\phi$$

$$dx\,dy\,dz = r^2\sin\phi\,dr\,d\phi\,d\theta$$

Given that $x^2+y^2+z^2=a^2$

Limits:

$r = 0$ to $r = a$

$\theta = 0$ to $\theta = 2\pi$

$\phi = 0$ to $\phi = \pi$

$$\text{Required volume} = \iiint_V dx\,dy\,dz$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{a} r^2\sin\phi\,dr\,d\phi\,d\theta$$

$$= \left[\int_0^a r^2\,dr\right]\left[\int_0^{\pi}\sin\phi\,d\phi\right]\left[\int_0^{2\pi} d\theta\right]$$

$$= \left[\frac{r^3}{3}\right]_0^a \left[-\cos\phi\right]_0^{\pi} \left[\theta\right]_0^{2\pi}$$

$$= \left[\frac{a^3}{3}\right]\left[-\cos\pi + \cos 0\right]\left[2\pi - 0\right]$$

$$= (a^3/3)\left[-(-1)+1\right]\left[2\pi\right]$$

$$= (a^3/3)\left[1+1\right]\left[2\pi\right] = (a^3/3)(2)(2\pi)$$

$$= \frac{4\pi a^3}{3}$$

MA3151-MATRICES AND CALCULUS

UNIT-2

DIFFERENTIAL CALCULUS

Mr.V.PRAKASH,M.Sc;M.Phil;B.Ed;
Assistant Professor
Department of Mathematics

UNIT-2 [DIFFERENTIAL CALCULUS]

CHAPTER-2.1 Representation of a function

Definition:-

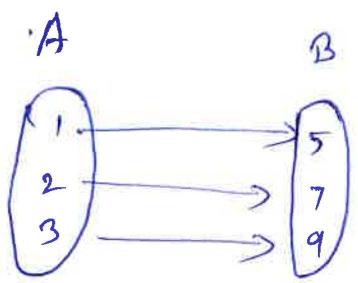
A function f is a rule that assigns to each element x in a set A. Exactly one element called $f(x)$ in a set B.

Example for function:-

The set $\{(1,5), (2,7), (3,9)\}$

Here, $A = \{1, 2, 3\}$ and $B = \{5, 7, 9\}$

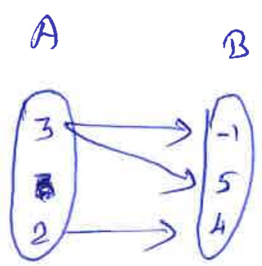
$f: A \longrightarrow B$



Example for not a function:-

The set $\{(3,-1), (3,5), (2,4)\}$

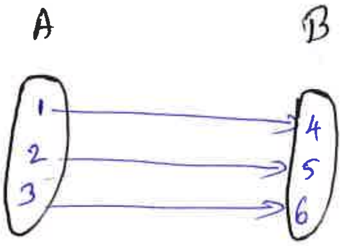
Here, $A = \{3, 3, 2\}$, $B = \{-1, 5, 4\}$



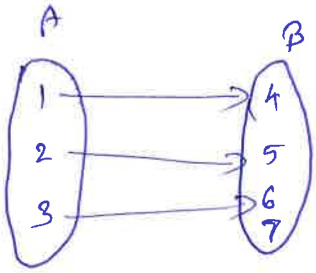
So, $f: A \longrightarrow B$ is not a function

Function	Non a Function

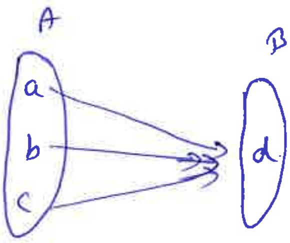
(i)



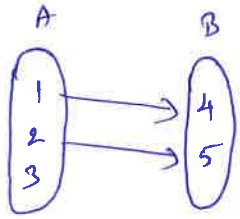
(ii)



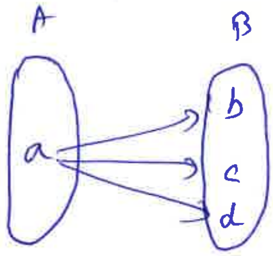
(iii)



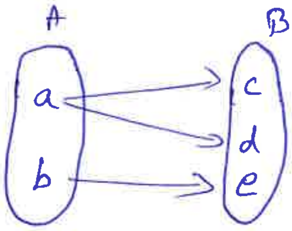
(i)



(ii)



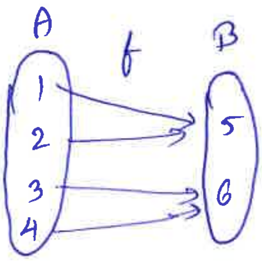
(iii)



TYPES OF FUNCTION :-

1) onto function

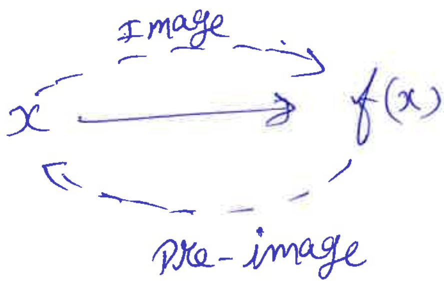
Eg: If $f: A \to B$, $f(A) = B$ (or) Range = domain



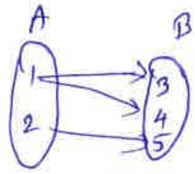
$f(A) = \{5, 6\}$

is $B = \{5, 6\}$

$\therefore f(A) = B$ is onto function



Eg:-

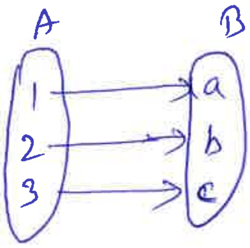


$f(A) \neq B$ is not onto function

2) <u>one to one functions</u>:-

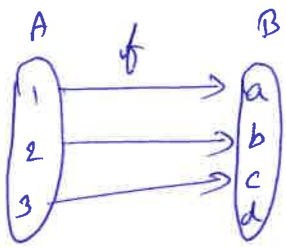
If $a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$, $a_1, a_2 \in A$

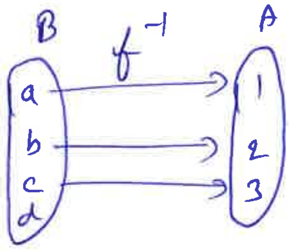
then $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$.



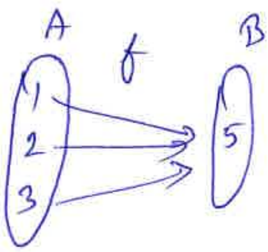
3) <u>Inverse function</u>:-

If $f: A \to B$ and $f^{-1}: B \to A$



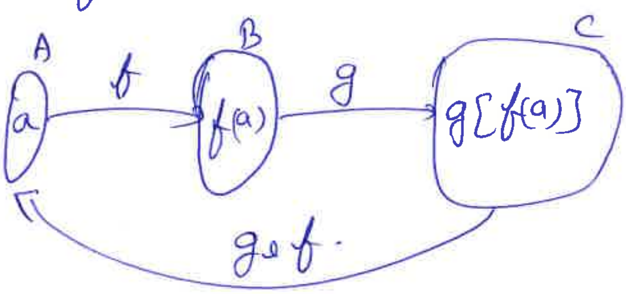


4) <u>Constant function</u>:-



5) <u>Composition function</u>:-

$f: A \to B$ and $g: B \to C$, then $g \circ f: A \to C$



Eg-① If $A = \{1, 2, 3\}$ defined by $f(x) = 2x+1$, then $f: A \to B$ is function or not.

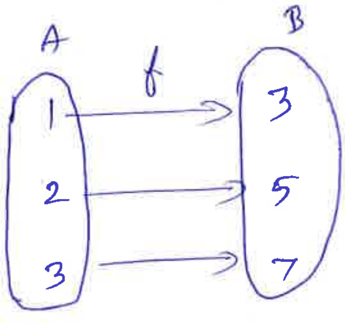
Soln: Given that $A = \{1, 2, 3\}$ and $B = \{3, 5, 7\}$

Define: $f(x) = 2x+1$

$f(1) = 2(1)+1 = 3$

$f(2) = 2(2)+1 = 5$

$f(3) = 2(3)+1 = 7$



is a function.

Eg-② Sketch the graph and find the domain and range of the following functions. (a) $f(x) = 3x-2$, (b) $g(x) = x^2$.

Soln:

(a) Let $y = 3x-2 \Rightarrow 3x = y+2$

$$x = \frac{y+2}{3}$$

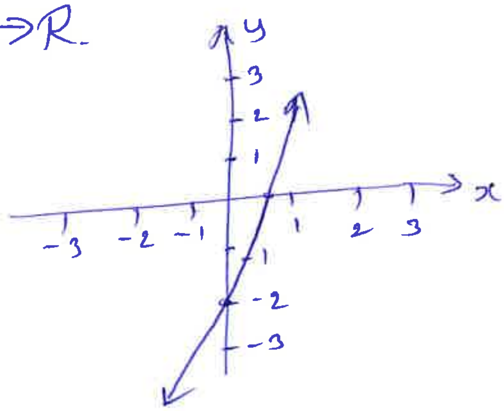
When $x = 0 \Rightarrow y = -2$

$y = 0 \Rightarrow x = 2/3$

$\therefore$ The coordinate pairs are $(0, -2)$ and $(2/3, 0)$.

Here, $y = 3x-2$ is defined for all real numbers. So the domain of f is the set of all real numbers denoted by R. and $x = \frac{y+2}{3}$ is defined for all real numbers.

So the range is R, $f: R \to R$.



Soln:

(b) Let $g(x) = x^2$

i.e, $y = x^2$

when $x = 0 \Rightarrow y = 0$

$x = 1 \Rightarrow y = 1$

$x = 2 \Rightarrow y = 4$

$x = 3 \Rightarrow y = 9$

$x = -1 \Rightarrow y = 1$

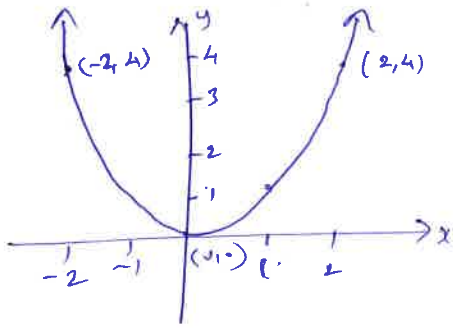
$x = -2 \Rightarrow y = 4$

$x = -3 \Rightarrow y = 9$

we get Plot the Points

$(0,0), (1,1), (2,4), (3,9)$

$(-1,1), (-2,4) (-3,9)$



The function $y = x^2$ is defined for all x, hence the domain is R.

and $y = x^2 \geq 0 \Rightarrow y \geq 0$

$\therefore$ The range of $g(x)$ is $\{y / y \geq 0\} = [0, \infty)$

Eg-(3) Find the domain of the functions (a) $f(x) = \sqrt{x+2}$

(b) $g(x) = \dfrac{1}{x^2 - x}$.

Soln.

(a) Let $y = \sqrt{x+2}$

Since, the square root of a negative number is not defined.

$\therefore x + 2 \geq 0 \Rightarrow x \geq -2$

hence the domain is $[-2, \infty)$

and $y = \sqrt{x+2} \geq 0$ if $x + 2 \geq 0$

i.e $y \geq 0$

hence the range is $[0, \infty)$

(b) Let $y = \dfrac{1}{x^2 - x}$

i.e, $y = \dfrac{1}{x(x-1)}$

Put $x(x-1) = 0 \Rightarrow x = 0, x = 1$

If $x = 0 \Rightarrow y$ is not Exist

If $x = 1 \Rightarrow y$ is not Exist

Hence the domain of $y = \{x / x \neq 0, x \neq 1\}$

i.e, $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$.

EVEN FUNCTION :-

If a function f satisfies $f(-x) = f(x)$ for every number x in its domain, then f is called an even function.

Eg: $f(x) = x^2$ is an even function

$$f(-x) = (-x)^2 = x^2 = f(x)$$

ODD FUNCTION :-

If a function f satisfies $f(-x) = -f(x)$ for every number x in its domain, then f is called an odd function.

Eg: $f(x) = x^3$ is an odd function

$$f(-x) = (-x)^3 = -x^3 = -f(x).$$

Increasing and Decreasing functions :-

A function f is called increasing on an interval I, if $f(x_1) < f(x_2)$, whenever $x_1 < x_2$ in I.

A function f is called decreasing on an interval I, if $f(x_1) > f(x_2)$, whenever $x_1 < x_2$ in I.

Eg - 1 Find the domain of the functions $f(x) = \frac{x+4}{x^2-9}$.

Soln:-

Given that $f(x) = \frac{x+4}{x^2-9}$

Put, $x^2-9=0 \Rightarrow x^2 = 9$.
$x = \pm 3$

If $x=3 \Rightarrow f(x) = \frac{3+4}{9-9} = \frac{7}{0} = \infty$

If $x=-3 \Rightarrow f(x) = \frac{-3+4}{9-9} = 1/0 = \infty$

The domain of f is $\{x / x \neq 3, x \neq -3\}$

which is written as $\{(-\infty, -3) \cup (-3, 3) \cup (3, \infty)\}$

Eg:- 2 Find the domain and range of functions $y = \sqrt{1-x^2}$

Soln:- Given $y = \sqrt{1-x^2}$

Since, square root of a negative number is not defined

$\therefore 1-x^2 \geq 0 \Rightarrow 1 \geq x^2$

$\Rightarrow x^2 \leq 1 \quad \therefore -1 \leq x \leq 1$

hence domain of f is $[-1, 1]$

If $x = -1 \Rightarrow y = 0$

If $x = 1 \Rightarrow y = 0$

If $x = 0 \Rightarrow y = 1$

hence the range of y is $[0, 1]$

Eg:- 3

Find the domain of the function $f(x) = \dfrac{2x^3-5}{x^2+x-6}$

Soln:

Given that $f(x) = \dfrac{2x^3-5}{x^2+x-6}$

$$f(x) = \frac{2x^3-5}{(x-2)(x+3)}$$

The function is undefined at $(x-2)(x+3)=0$

$x-2=0$ | $x+3=0$

$\boxed{x=2}$ | $\boxed{x=-3}$

$\therefore$ The domain of f is $\{x / x \neq 2, x \neq -3\}$

which is written as $(-\infty,-3) \cup (-3,2) \cup (2,\infty)$.

Eg - 4 Evaluate the difference quotient of the function

$f(x) = 4+3x-x^2$, $\dfrac{f(3+h)-f(3)}{h}$.

Soln:

Let $f(x) = 4+3x-x^2$

$$f(3+h) = 4+3(3+h)-(3+h)^2$$

$$= 4+9+3h-[3^2+h^2+6h]$$

$$= 4+9+3h-3^2-h^2-6h$$

$$f(3+h) = 4-3h-h^2$$

$$f(3) = 4+3(3)-3^2 = 4+9-9 = 4$$

$$\therefore \frac{f(3+h)-f(3)}{h} = \frac{\cancel{4}-3h-h^2-\cancel{4}}{h} = \frac{-3h-h^2}{h} = \frac{\cancel{h}(-3-h)}{\cancel{h}}$$

$$= -3-h$$

CHAPTER - 2.2 Limit of a function

Defn Suppose $f(x)$ is defined when x is near the number a, then the limit of that function is

$$\lim_{x \to a} f(x) = L.$$

ie, $f(x) \longrightarrow L$ as $x \longrightarrow a$.

Left-hand Limit :-

$$\lim_{x \to a^-} f(x) = L.$$

Right-hand Limit :-

$$\lim_{x \to a^+} f(x) = L.$$

Note :- $\lim_{x \to a} f(x) = L \iff \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$

Limit Rules :-

① $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$

② $\lim_{x \to a} [c\, f(x)] = c \lim_{x \to a} f(x)$

③ $\lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

④ $\lim_{x \to a} \left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, if $\lim_{x \to a} g(x) \neq 0$

(5) $\lim_{x \to a} c = c$

(6) $\lim_{x \to a} x = a$

(7) $\lim_{x \to a} x^n = a^n$

(8) $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$

(9) $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$, n is +ve

Eg - (1) Find the value of $\lim_{x \to 1} \frac{x-1}{x^2-1}$

Soln:- Let $f(x) = \frac{x-1}{x^2-1}$

This function is not defined at $x=1$.

$x < 1$	$f(x)$	$x > 1$	$f(x)$
0.5	0.6667	1.5	0.4
0.9	0.5263	1.1	0.4762
0.99	0.5025	1.01	0.4975
0.999	0.5002	1.001	0.4998
0.9999	0.5	1.0001	0.5

$\therefore \lim_{x \to 1} \frac{x-1}{x^2-1} = 0.5$

Eg - 2 Find the value of $\lim_{t \to 0} \frac{\sqrt{t^2+9}-3}{t^2}$

Soln:-

Let $f(t) = \frac{\sqrt{t^2+9}-3}{t^2}$

t	$f(t)$
± 0.5	0.1655
± 0.1	0.1666
± 0.05	0.1667
± 0.01	0.1667
± 0.001	0.1667
± 0.0001	0.1667
± 0.00001	0.167

$$\therefore \lim_{t \to 0} \frac{\sqrt{t^2+9}-3}{t^2} = \cancel{0.167}\ 0.167$$

Eg - 3 Find the value of $\lim_{x \to 0} 1/x^2$ if it exists.

Soln:-

Let $f(x) = 1/x^2$

x	$f(x)$
± 1	1
± 0.5	4
± 0.2	25
± 0.1	100
± 0.01	$10,000$

$$\lim_{x \to 0} 1/x^2 = \infty.$$

Eg -4 Find the value of $\lim_{x\to 3^+} \frac{2x}{x-3}$ and $\lim_{x\to 3^-} \frac{2x}{x-3}$

Soln:

Given $\lim_{x\to 3^-} \frac{2x}{x-3}$ and $\lim_{x\to 3^+} \frac{2x}{x-3}$

x	$f(x)$	x	$f(x)$
2.9	-58	3.01	602
2.99	-598	3.001	6002
2.999	-5998	3.000001	6000002
2.9999	-5999998	3.00000001	600000002

from the above takes

$$\lim_{x\to 3^+} \frac{2x}{x-3} = \infty, \quad \lim_{x\to 3^-} \frac{2x}{x-3} = \infty$$

Eg-5 Evaluate the given Limit $\lim_{x\to 5} (2x^2-3x+4)$

Soln:

Given, $\lim_{x\to 5} (2x^2-3x+4) = \lim_{x\to 5} 2x^2 - \lim_{x\to 5} 3x + \lim_{x\to 5} 4$

$= 2(5)^2 - 3(5) + 4$

$= 2(25) - 15 + 4$

$= 50 - 15 + 4$

$= 39.$

Eg - 6 Find the value of $\lim\limits_{x \to -2} \dfrac{x^3+2x^2-1}{5-3x}$.

Sol^n:

Given $\lim\limits_{x \to -2} \dfrac{x^3+2x^2-1}{5-3x}$

$$= \frac{\lim\limits_{x \to -2} (x^3+2x^2-1)}{\lim\limits_{x \to -2} (5-3x)}$$

$$= \frac{(-2)^3+2(-2)^2-1}{5-3(-2)} = \frac{-8+2(4)-1}{5+6} = \frac{-8+8-1}{11}$$

$$= -1/11$$

Eg - 7 Show that $\lim\limits_{x \to 0} |x| = 0$.

Sol^n:

Given that $|x| = \begin{cases} x, & \text{if } x > 0 \\ -x, & \text{if } x < 0 \end{cases}$

$\therefore \lim\limits_{x \to 0^+} |x| = \lim\limits_{x \to 0^+} x = 0$

$\lim\limits_{x \to 0^-} |x| = \lim\limits_{x \to 0^-} (-x) = 0$

$\therefore$ We have $\lim\limits_{x \to 0^+} |x| = \lim\limits_{x \to 0^-} |x| = 0$

$\therefore \lim\limits_{x \to 0} |x| = 0$

Eg - 8 Prove that $\lim_{x\to 0} \frac{|x|}{x}$ does not exist.

soln:

$$\lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} \frac{x}{x} = \lim_{x\to 0^+} \frac{1}{1} = 1$$

$$\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = \lim_{x\to 0^-} \frac{-1}{1} = -1$$

$$\therefore \lim_{x\to 0^+} \frac{|x|}{x} \neq \lim_{x\to 0^-} \frac{|x|}{x}$$

$\therefore \lim_{x\to 0} \frac{|x|}{x}$ is not exist

The squeeze theorem:-

If $f(x) \le g(x) \le h(x)$ when x is near 'a'

and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.

Eg - 9 Show that $\lim_{x\to 0} x^2 \sin(1/x) = 0$ by using squeeze thm.

soln:-

Here we cannot use

$$\lim_{x\to 0} x^2 \sin(1/x) = \lim_{x\to 0} x^2 \lim_{x\to 0} \sin(1/x)$$

Because $\lim_{x\to 0} \sin(1/x)$ does not exist

W.K.T $-1 \le \sin x \le 1 \quad \forall x$

$$-1 \le \sin 1/x \le 1$$

$$-x^2 \le x^2 \sin 1/x \le x^2$$

$$\Rightarrow \lim_{x\to 0} (-x^2) \le \lim_{x\to 0} x^2 \sin(1/x) \le \lim_{x\to 0} x^2$$

$$0 \le \lim_{x\to 0} x^2 \sin(1/x) \le 0$$

$$\lim_{x\to 0} x^2 \sin(1/x) = 0.$$

Eg - (10) Evaluate $\lim_{x \to \pi/2} \frac{1+\cos 2x}{(\pi-2x)^2}$

Soln:

$$\lim_{x \to \pi/2} \frac{1+\cos 2x}{(\pi-2x)^2} = \lim_{x \to \pi/2} \frac{2\cos^2 x}{(\pi-2x)^2}$$

$$= \lim_{x \to \pi/2} \frac{2\sin^2(\pi/2 - x)}{4(\pi/2 - x)^2}$$

$$= \lim_{x \to \pi/2} \frac{1}{2}\left[\frac{\sin(x-\pi/2)}{(\pi/2 - x)}\right]^2$$

Let $\theta = x - \pi/2$, we get

$$= \lim_{x \to 0} \frac{1}{2}\left[\frac{\sin\theta}{\theta}\right]^2$$

$$= \frac{1}{2}\left[\lim_{x \to 0} \frac{\sin\theta}{\theta}\right]^2 = \frac{1}{2}(1)^2 = \frac{1}{2}$$

Continuity

Example-(13) (uni-case) Find the values of a and b that makes f continuous on $(-\infty,\infty)$ $f(x)=\begin{cases}\frac{x^3-8}{x-2}, & \text{if } x<2\\ ax^2-bx+3, & \text{if } 2\le x<3\\ 2x-a+b, & \text{if } x\ge 3\end{cases}$

Soln

Let $f(x)=\begin{cases}\frac{x^3-8}{x-2}, & \text{if } x<2\\ ax^2-bx+3, & \text{if } 2\le x<3\\ 2x-a+b, & \text{if } x\ge 3\end{cases}$

The given function f(x) is continuous on $(-\infty,2)$, $(2,3)$, $(3,\infty)$

Now (i) $\underset{x\to 2^-}{Lt}\ \frac{x^3-8}{x-2}=\frac{(2)^3-8}{2-2}=\frac{0}{0}=\infty$ {using L-hospital rule}

$\underset{x\to 2^-}{Lt}\ \frac{3x^2}{1}=\frac{3(2)^2}{1}=3(4)=12 \longrightarrow (1)$

(ii) $\underset{x\to 2^+}{Lt}\ ax^2-bx+3=a(2)^2-b(2)+3=4a-2b+3 \longrightarrow (2)$

$\underset{x\to 3^-}{Lt}\ ax^2-bx+3=a(3)^2-b(3)+3=9a-3b+3 \longrightarrow (3)$

$\underset{x\to 3^+}{Lt}\ 2x-a+b=2(3)-a+b=6-a+b \longrightarrow (4)$

from (1) & (2)

$12=4a-2b+3$

$4a-2b=12-3$

$4a-2b=9 \longrightarrow (5)$

from (3) & (4)

$9a-3b+3=6-a+b$

$9a+a-3b-b=6-3$

$10a-4b=3 \longrightarrow (6)$

from (5) & (6)

$(5)\times 2 \Rightarrow\ 8a-\cancel{4b}=18$

$(6) \Rightarrow\ \underset{(-)}{10a}\ \underset{(+)}{-\cancel{4b}}=\underset{(-)}{3}$

$-2a=15$

$\boxed{a=\frac{-15}{2}}$

$(5)\Rightarrow\ 4a-2b=9$

$2\cancel{4}\left(-15/\cancel{2}\right)-2b=9$

$2(-15)-2b=9$

$-30-2b=9$

$-2b=9+30$

$-2b=39$

$\boxed{b=-39/2}$

The value of $\boxed{a=-\frac{15}{2}}$ and $\boxed{b=\frac{-39}{2}}$ //

CONTINUITY :-

Defn: A function f is continuous at a number 'a' if $\lim_{x\to a} f(x) = f(a)$.

The above definition implicitly requires three points.

(i) $f(a)$ is defined

(ii) $\lim_{x\to a} f(x)$ exists, (iii) $\lim_{x\to a} f(x) = f(a)$.

Eg-① Where are each of the following functions discontinuous? (a) $f(x) = \dfrac{x^2-x-2}{x-2}$ (b) $f(x) = \begin{cases} 1/x^2, & x \neq 0 \\ 1, & x = 0 \end{cases}$

(c) $f(x) = \begin{cases} \dfrac{x^2-x-2}{x-2}, & x \neq 2 \\ 1, & x = 2. \end{cases}$

Soln:-

(a) Given $f(x) = \dfrac{x^2-x-2}{x-2}$

Put $x-2=0 \Rightarrow x=2$

$\therefore$ Hence f is discontinuous at $x=2$

(b) Given $f(x) = \begin{cases} 1/x^2, & x \neq 0 \\ 1, & x = 0 \end{cases}$

Here, $f(0) = 1$, but $\lim_{x\to 0} 1/x^2$ not exist

So, f is discontinuous at $x=0$.

(c) Given $f(x) = \begin{cases} \dfrac{x^2-x-2}{x-2}, & x \neq 2 \\ 1, & x = 2 \end{cases}$

Here, $f(2) = 1$ is defined and

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^2-x-2}{x-2} = \lim_{x \to 2} \frac{\cancel{(x-2)}(x+1)}{(x-2)}$$

$$= \lim_{x \to 2} (x+1)$$

$$= 2+1$$

$$= 3$$

$$\neq f(2)$$

So, f is continuous at '2'

<u>Eg - (8)</u> Show that the function $f(x) = 1 - \sqrt{1-x^2}$ is continuous in the interval $[-1, 1]$.

<u>Soln:</u>

Consider $\lim_{x \to a} f(x) = \lim_{x \to a} \left[1 - \sqrt{1-x^2}\right]$ if $-1 < a < 1$

$$= 1 - \sqrt{\lim_{x \to a} (1-x^2)}$$

$$= 1 - \sqrt{1-a^2}$$

$$= f(a)$$

$\therefore$ f is continuous at 'a' if $-1 < a < 1$

Also $\lim_{x \to -1^+} f(x) = 1 = f(-1)$

$\lim_{x \to 1^-} f(x) = 1 = f(1)$

$\therefore$ f is continuous from the right at -1 and left at 1. $\therefore$ f is continuous on $[-1, 1]$.

Example-③ Find $\underset{x\to -2}{Lt} \frac{x^3+2x^2-1}{5-3x}$.

Sol^n:

Let us consider the function $f(x) = \frac{x^3+2x^2-1}{5-3x}$ is rational. By known theorem it is continuous on its domain.

which is $\{x / x \neq 5/3\}$

$$\therefore \underset{x\to -2}{Lt} \frac{x^3+2x^2-1}{5-3x} = \lim_{x\to -2} f(x) = f(-2)$$

$$= \frac{(-2)^3+2(-2)^2-1}{5-3(-2)}$$

$$= \frac{-8+2(4)-1}{5+6} = \frac{\cancel{-8}+\cancel{8}-1}{11}$$

$$= -1/11$$

Example-④ Show that the function is continuous at the number 'a' by using definition of continuity and properties of limits.

$f(x) = (x+2x^3)^4$, $a=-1$.

Sol^n:

Given that $f(x) = (x+2x^3)^4$, $a=-1$

W.K.T $\underset{x\to a}{Lt} f(x) = \underset{x\to -1}{Lt} (x+2x^3)^4$

$$= \left[\underset{x\to -1}{Lt} (x+2x^3)\right]^4$$

$$= [-1+2(-1)^3]^4 = [-1+2(-1)]^4$$

$$= (-1-2)^4 = (-3)^4$$

$$= 81$$

Example-5 For what value of the constant c is the function f continuous on $(-\infty,\infty)$. $f(x)=\begin{cases} cx^2+2x, & x<2 \\ x^3-cx, & x\geq 2 \end{cases}$

Soln:

Given that $f(x)=\begin{cases} cx^2+2x, & x<2 \\ x^3-cx, & x\geq 2 \end{cases}$ is continuous on $(-\infty,2)$ & $(2,\infty)$

Now, $\underset{x\to 2^-}{Lt} f(x) = \lim_{x\to 2^-}[cx^2+2x]$

$= c(2)^2+2(2) = 4c+4 \longrightarrow (1)$

$\underset{x\to 2^+}{Lt} f(x) = \lim_{x\to 2^+}[x^3-cx]$

$= (2)^3-c(2) = 8-2c \longrightarrow (2)$

from (1) & (2), f is continuous at $x=2$.

$\therefore 4c+4 = 8-2c$

$4c+2c = 8-4$

$6c = 4$

$c = 4/6$

$\boxed{c = 2/3}$

$\therefore$ f is continuous on $(-\infty,\infty)$ for $c=2/3$.

Example-6 Show that the function is continuous at the number 'a' by using definition of continuity and properties.

$f(x) = 3x^4-5x+\sqrt[3]{x^2+4}, \quad a=2.$

Soln: Given that $f(x) = 3x^4-5x+\sqrt[3]{x^2+4}, \quad a=2.$

$$\lim_{x \to a} f(x) = \lim_{x \to 2} \left[3x^4 - 5x + \sqrt[3]{x^2+4}\right]$$

$$= 3 \lim_{x \to 2} x^4 - 5 \lim_{x \to 2} x + \lim_{x \to 2} \sqrt[3]{x^2+4}$$

$$= 3(2)^4 - 5(2) + \sqrt[3]{(2)^2+4}$$

$$= 3(16) - 10 + \sqrt[3]{4+4}$$

$$= 48 - 10 + \sqrt[3]{8}$$

$$= 48 - 10 + 2$$

$$f(2) = 40$$

By the definition of continuity, f is continuous at $a=2$.

Example-(7)

(i) Find $\lim_{x \to 4} \dfrac{5+\sqrt{x}}{\sqrt{5+x}}$.

Soln.

Given that $\underset{x \to 4}{Lt} \dfrac{5+\sqrt{x}}{\sqrt{5+x}}$

$$= \frac{5+\sqrt{4}}{\sqrt{5+4}} = \frac{5+2}{\sqrt{9}} = \frac{7}{3} //$$

(ii) Find $\lim_{x \to 1} e^{x^2-x}$

Soln:

Given that $\underset{x \to 1}{Lt} e^{x^2-x}$

$$= e^{(1)^2-1} = e^{1-1} = e^0$$

$$= 1 //$$

Example - 8 Discuss the continuity of $f(x) = \tan x$.

Soln:

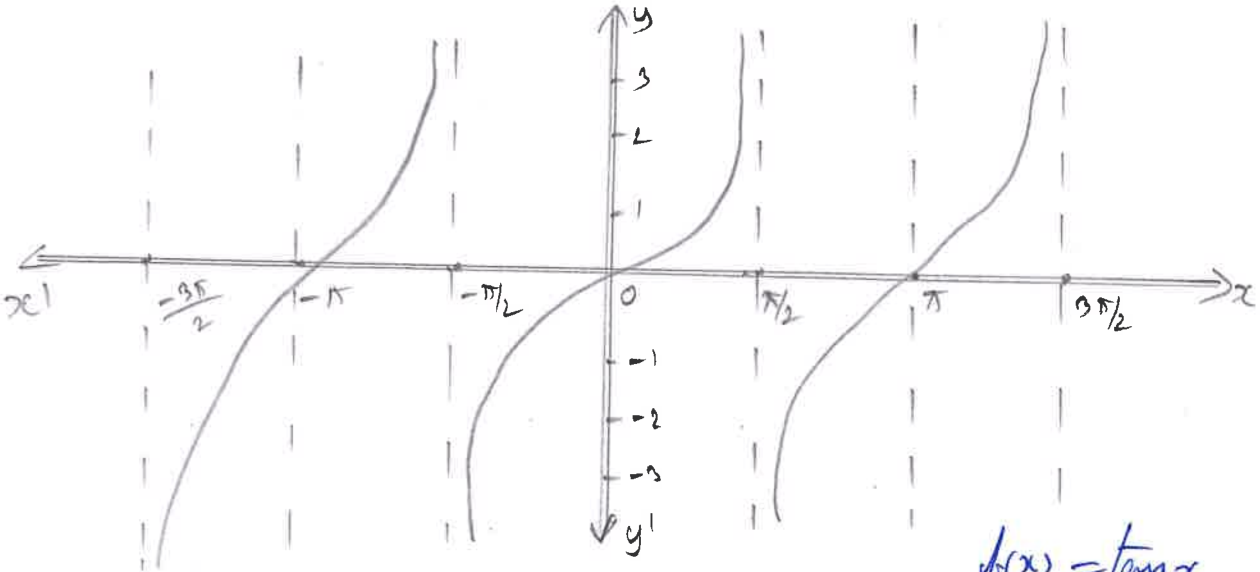
Given that $f(x) = \tan x$.

$\tan x = \frac{\sin x}{\cos x}$ is continuous except $\cos x = 0$.

This happens when x is odd integer multiple of $\pi/2$.

So $y = \tan x$ has infinite discontinuities.

when $x = \pm \pi/2, \pm 3\pi/2, \pm 5/2, \ldots$ and so on.



Example - 9 Evaluate $\lim\limits_{x \to \pi} \frac{\sin x}{2 + \cos x}$.

Soln:

W.K.T $y = \sin x$ is continuous. The function in the denominator $y = 2 + \cos x$ is the sum of two continuous functions. This function is never zero because $\cos x \geq -1$. $\forall x$. and $2 + \cos x \geq 0$. $\therefore$ The ratio $f(x) = \frac{\sin x}{2 + \cos x}$ is continuous everywhere. By defn.

$$\lim_{x \to \pi} f(x) = \lim_{x \to \pi} \frac{\sin x}{2 + \cos x} = \frac{\sin \pi}{2 + \cos \pi} = \frac{0}{2+1} = 0$$

$$\therefore f(\pi) = 0$$

The derivative of a function f at a number 'a', denoted by $f'(a)$ is $f'(a) = \frac{f(a+h) - f(a)}{h}$ if this Limit Exists.

If we write $x = a+h$, then we take $h = x - a$ and h approaches '0' iff x approaches 'a'.

$$\therefore f'(a) = \lim_{x \to a} \left[\frac{f(x) - f(a)}{x - a} \right].$$

Example-① Find the Equation of the tangent line to the Parabola $y = x^2$ at the point $P(1,1)$.

Soln:- Here we have $a = 1$ and $f(x) = x^2$.

So the slope is $m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

$$= \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

$$= \lim_{x \to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}}$$

$$= \lim_{x \to 1} (x+1) = 1+1 = 2.$$

W.K.T Eqn of tangent line at (x_1, y_1) is

$(y - y_1) = m(x - x_1)$, Here $(x_1, y_1) = (1,1)$

$(y - 1) = 2(x - 1)$

$y - 1 = 2x - 2$

$y = 2x - 2 + 1$

$y = 2x - 1$ //

Example - 2 Find the Equation of the tangent line to the hyperbola $y = 3/x$ at the Point $(3,1)$.

soln:-

Let us Consider $f(x) = 3/x$.

then the slope of the Point at $(3,1)$ is

$$m = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}, \quad \text{Here } a=3$$

$$= \lim_{h\to 0} \frac{f(3+h)-f(3)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{3}{3+h} - 1}{h} = \lim_{h\to 0} \frac{\left(\frac{3-(3+h)}{3+h}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\left(\frac{\cancel{3}-\cancel{3}-h}{3+h}\right)}{h} = \lim_{h\to 0} \frac{-\cancel{h}}{\cancel{h}(3+h)}$$

$$= \lim_{h\to 0} \frac{-1}{3+h} = \frac{-1}{3+0} = -1/3$$

$$\boxed{m = -1/3}$$

The Eqn of the tangent at the Point $(3,1)$ is

$$(y-y_1) = m(x-x_1)$$

$$y-1 = -1/3\,(x-3)$$

$$3(y-1) = -(x-3)$$

$$3y-3 = -x+3$$

$$x+3y-3-3 = 0$$

$$x+3y-6 = 0.$$

Example - 3

Find the tangent line to the Equation $x^3+y^3=6xy$ at the Point (3,3) and at what Point the tangent line horizontal in the first quadrant.

Soln:-

Given that $x^3+y^3=6xy$

To find (slope) $m = dy/dx$

i.e., $x^3+y^3=6xy$

$$\frac{d}{dx}(x^3+y^3) = \frac{d}{dx}(6xy)$$

$$3x^2+3y^2\,dy/dx = 6x\,dy/dx + 6y$$

$$3y^2\,dy/dx - 6x\,dy/dx = 6y-3x^2$$

$$dy/dx\,[3y^2-6x] = 3(2y-x^2)$$

$$dy/dx = \frac{3(2y-x^2)}{3y^2-6x}$$

$$\text{slope} = m = dy/dx = \frac{\cancel{3}(2y-x^2)}{\cancel{3}(y^2-2x)} = \frac{2y-x^2}{y^2-2x}$$

$$\left(\frac{dy}{dx}\right)_{at\ (3,3)} = \frac{2(3)-(3)^2}{(3)^2-2(3)} = \frac{6-9}{9-6} = \frac{-\cancel{3}}{\cancel{3}}$$

$$\boxed{m=-1}$$

The Equation of the tangent at the Point (3,3) is

$$(y-y_1) = m(x-x_1)$$

$$(y-3) = -1(x-3)$$

$$y-3 = -x+3$$

$$y+x = 3+3$$

$$\boxed{x+y=6}$$

The tangent line is $\boxed{x+y=6}$.

i.e, $y = 6-x$

To find horizontal tangent :-

The curve $y = 6-x$.

$$\frac{dy}{dx} = 0 \Rightarrow 0-1=0 \Rightarrow$$

$\therefore$ The curve $y=6-x$ has horizontal tangent at $x =$

Example - 4

Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so where?

Soln:

Given $y = x^4 - 2x^2 + 2$.

To find horizontal tangent :-

If any occur where $dy/dx = 0$.

To find these points. i.e, $y = x^4 - 2x^2 + 2$

$$dy/dx = 4x^3 - 4x.$$

$\Rightarrow 4x^3 - 4x = 0$

$4x(x^2-1) = 0 \Rightarrow 4x = 0$ and $x^2 - 1 = 0$

$\boxed{x=0}$ and $x^2 = 1 \Rightarrow \boxed{x = \pm 1}$

$\therefore$ The horizontal tangents at $x = 0, 1, -1$.

Example - 5 If $f(x) = \frac{1-x}{2+x}$ then, find the Equation for $f'(x)$ using the Concept of derivatives.

Sol/w Given that $f(x) = \frac{1-x}{2+x}$.

W.K.T $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h\to 0} \left[\frac{\frac{1-(x+h)}{2+(x+h)} - \frac{1-x}{2+x}}{h} \right]$$

$$= \lim_{h\to 0} \left[\frac{\frac{(1-x-h)}{(2+x+h)} - \frac{(1-x)}{(2+x)}}{h} \right]$$

$$= \lim_{h\to 0} \left[\frac{\frac{(2+x)(1-x-h) - (1-x)(2+x+h)}{(2+x)(2+x+h)}}{h} \right]$$

$$= \lim_{h\to 0} \left[\frac{(2-2x-2h+x-x^2-hx) - (2+x+h-2x-x^2-hx)}{h(2+x)(2+x+h)} \right]$$

$$= \lim_{h\to 0} \left[\frac{\cancel{2}-x-\cancel{x^2}-2h-\cancel{hx}-\cancel{2}-x-h+2x+\cancel{x^2}+\cancel{hx}}{h(2+x)(2+x+h)} \right]$$

$$= \lim_{h\to 0} \left[\frac{-\cancel{2x}+\cancel{2x}-3h}{h(2+x)(2+x+h)} \right]$$

$$= \lim_{h\to 0} \left[\frac{-3\cancel{h}}{\cancel{h}(2+x)(2+x+h)} \right]$$

$$= \underset{h\to 0}{Lt}\left[\frac{-3}{(2+x)(2+x+h)}\right]$$

$$= \frac{-3}{(2+x)(2+x+0)} = \frac{-3}{(2+x)(2+x)}$$

$$f'(x) = \frac{-3}{(2+x)^2}.$$

Example - 6 If $f(x) = x^3 - x$, then find the Equ $f'(x)$. using the properties.

Sol/n Given that $f(x) = x^3 - x$.

W.K.T $f'(x) = \underset{h\to 0}{Lt} \dfrac{f(x+h) - f(x)}{h}$

$$= \underset{h\to 0}{Lt}\left[\frac{(x+h)^3 - (x+h) - (x^3 - x)}{h}\right]$$

$$= \underset{h\to 0}{Lt}\left[\frac{(x^3+3x^2h+3xh^2+h^3) - (x+h) - x^3 + x}{h}\right]$$

$$= \underset{h\to 0}{Lt}\left[\frac{\cancel{x^3}+3x^2h+3xh^2+h^3-\cancel{x}-h-\cancel{x^3}+\cancel{x}}{h}\right]$$

$$= \underset{h\to 0}{Lt}\left[\frac{3x^2h+3xh^2+h^3-h}{h}\right]$$

$$= \underset{h\to 0}{Lt}\left[\frac{\cancel{h}[3x^2+3xh+h^2-1]}{\cancel{h}}\right]$$

$$= \underset{h\to 0}{Lt}\left[3x^2+3xh+h^2-1\right]$$

$$= 3x^2 + (0) + (0) - 1$$

$$f'(x) = 3x^2 - 1$$

Example ⑦ Determine whether $f'(0)$ exists or not for the given function $f(x) = \begin{cases} x \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$

Soln:

Since, $f(x) = x \sin(1/x)$, $x \neq 0$

and $f(x) = 0$, when $x = 0$

We have
$$f'(x) = \underset{h \to 0}{Lt} \frac{f(x+h) - f(x)}{h}$$

$$f'(0) = \underset{h \to 0}{Lt} \frac{f(0+h) - f(0)}{h}$$

$$= \underset{h \to 0}{Lt} \frac{f(h) - f(0)}{h}$$

$$= \underset{h \to 0}{Lt} \frac{h \sin(1/h) - 0}{h}$$

$$= \underset{h \to 0}{Lt} \frac{\cancel{h} \sin(1/h)}{\cancel{h}}$$

$$f'(0) = \underset{h \to 0}{Lt} \sin(1/h)$$

This limit does not exist. since $\sin(1/h)$ assumes the values -1 and 1 on any interval having 0.

$\therefore$ $f'(0)$ does not exist.

Example :- 8

If $f(x) = 2x^3 + x$ then find the Equ for $f'(a)$. Using the Properties.

Soln: Given that $f(x) = 2x^3 + x$.

W.K.T $f'(x) = \underset{h \to 0}{Lt} \dfrac{f(x+h) - f(x)}{h}$

$$= \underset{h \to 0}{Lt} \left[\frac{2(x+h)^3 + (x+h) - (2x^3 + x)}{h} \right]$$

$$= \underset{h \to 0}{Lt} \left[\frac{2(x^3 + 3x^2h + 3xh^2 + h^3) + \cancel{x} + h - 2x^3 - \cancel{x}}{h} \right]$$

$$= \underset{h \to 0}{Lt} \left[\frac{\cancel{2x^3} + 6x^2h + 6xh^2 + 2h^3 + h - \cancel{2x^3}}{h} \right]$$

$$= \underset{h \to 0}{Lt} \left[\frac{6x^2h + 6xh^2 + 2h^3 + h}{h} \right]$$

$$= \underset{h \to 0}{Lt} \left[\frac{\cancel{h}(6x^2 + 6xh + 2h^2 + 1)}{\cancel{h}} \right]$$

$$= \underset{h \to 0}{Lt} (6x^2 + 6xh + 2h^2 + 1)$$

$$= 6x^2 + (0) + (0) + 1$$

$f'(x) = 6x^2 + 1$

$f'(a) = 6a^2 + 1$ //

Eg - 1 If $y = (x^3-1)^{100}$, find $dy/dx = ?$

Soln.

Given $y = (x^3-1)^{100}$

$$\frac{dy}{dx} = 100(x^3-1)^{99}(3x^2)$$

$$\frac{dy}{dx} = 300x^2(x^3-1)^{99}$$

Eg - 2 If $x^2+y^2=25$, then find dy/dx

Soln.

Given $x^2+y^2=25$

$$d/dx(x^2+y^2) = d/dx(25)$$

$$2x + 2y\, dy/dx = 0$$

$$2y\, dy/dx = -2x$$

$$dy/dx = \frac{-\cancel{2}x}{\cancel{2}y}$$

$$dy/dx = -x/y$$

Eg - 3 Find dy/dx if $x^3+y^3=6xy$.

Soln.

Given $x^3+y^3 = 6xy$

$$d/dx(x^3+y^3) = d/dx(6xy)$$

$$3x^2+3y^2\, dy/dx = 6x\, dy/dx + 6y$$

$$3x^2+3y^2\, dy/dx - 6x\, dy/dx = 6y$$

$$3y^2\, dy/dx - 6x\, dy/dx = 6y-3x^2$$

$$dy/dx \cdot (3y^2-6x) = 6y-3x^2$$

$$dy/dx = \frac{6y-3x^2}{3y^2-6x}$$

Eg - 5 Show that the sum of x and y-intercepts of any tangent line to the curve $\sqrt{x}+\sqrt{y}=\sqrt{c}$ is Equal to c.

Soln:

Given $\sqrt{x}+\sqrt{y}=\sqrt{c}$

Diff. w.r.to 'x' on both sides

$$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\,dy/dx=0$$

$$\frac{1}{2\sqrt{y}}\,dy/dx=-\frac{1}{2\sqrt{x}}$$

$$dy/dx=\frac{-1}{\cancel{2}\sqrt{x}}\cdot\cancel{2}\sqrt{y}$$

$$dy/dx=\frac{-\sqrt{y}}{\sqrt{x}}$$

Let (a,b) be a point on the curve

$$\text{slope } m=dy/dx\,(a,b)=\frac{-\sqrt{b}}{\sqrt{a}}=-\sqrt{b/a}$$

Eqn of the tangent is

$$(y-y_1)=m(x-x_1)$$

$$(y-b)=-\sqrt{b/a}\,(x-a)$$

$$\frac{y-b}{x-a}=-\sqrt{b/a}$$

$$\sqrt{a}\,(y-b)=-\sqrt{b}\,(x-a)$$

$$y\sqrt{a}-b\sqrt{a}=-x\sqrt{b}+a\sqrt{b}$$

$$x\sqrt{b}+y\sqrt{a}=a\sqrt{b}+b\sqrt{a}$$

$$\frac{x\sqrt{b}}{a\sqrt{b}+b\sqrt{a}}+\frac{y\sqrt{a}}{a\sqrt{b}+b\sqrt{a}}=1$$

$$\frac{x}{a+\sqrt{ab}}+\frac{y}{b+\sqrt{ab}}=1$$

here, x - intercept is $a+\sqrt{ab}$ and y - intercept is $b+\sqrt{ab}$

$\therefore$ sum of the intercepts is

$$a+\sqrt{ab}+b+\sqrt{ab} = a+b+2\sqrt{ab}$$
$$= (\sqrt{a}+\sqrt{b})^2$$
$$= (\sqrt{c})^2$$
$$= c$$

$\left\{ \because (a,b) \text{ Point on } \sqrt{x}+\sqrt{y}=\sqrt{c} \right.$
$\sqrt{a}+\sqrt{b}=\sqrt{c}$.

Eg - 5 Find the first two derivatives for $x^4+y^4=16$.

Soln:

Given that $x^4+y^4=16 \longrightarrow (1)$

Diff (1). w.r.to. x:

$$4x^3+4y^3\, dy/dx = 0$$
$$4y^3\, dy/dx = -4x^3$$
$$y^3\, dy/dx = -x^3$$
$$dy/dx = \frac{-x^3}{y^3}$$
$$\boxed{y' = \frac{-x^3}{y^3}} \longrightarrow (2)$$

Again Diff (2). w.r.to. x. $\left\{ \text{W.k.T } \frac{u}{v} = \frac{v\,du - u\,dv}{v^2} \right\}$

Here, $u = x^3$ | $v = y^3$

$du = 3x^2$ | $dv = 3y^2\, dy/dx$.

$dv = 3y^2 y'$

$$\frac{u}{v} = \frac{v\,du - u\,dv}{v^2}$$

$$y'' = -\left[\frac{y^3(3x^2) - x^3(3y^2 y')}{(y^3)^2}\right]$$

$$= -\left[\frac{3y^3x^2 - x^3\left[3\cancel{y^2}\left(-\frac{x^3}{\cancel{y^3}}\right)\right]}{y^6}\right]$$

$$= -\left[\frac{3x^2y^3 - x^3\left[\frac{-3x^3}{y}\right]}{y^6}\right]$$

$$= -\left[\frac{3x^2y^3 + \frac{3x^6}{y}}{y^6}\right]$$

$$= -\left[\frac{3x^2y^4 + 3x^6}{y^7}\right] = -\left[\frac{3x^2(y^4+x^4)}{y^7}\right]$$

$$= -\left[\frac{3x^2(16)}{y^7}\right] = -\frac{48x^2}{y^7}$$

[illegible]

$$\left\{x^4+y^4=16\right\}$$

$$y'' = \frac{-48x^2}{y^7} \quad //$$

Eg - 6 If $f(x) = xe^x$, find $f'(x)$.

Soln:

Given $f(x) = xe^x$

$$f'(x) = d/dx[xe^x]$$
$$= xe^x + e^x(1)$$
$$= xe^x + e^x$$
$$= e^x[x+1]$$

Eg - 7 If $f(x) = \dfrac{x^2+x-2}{x^3+6}$. find $f'(x)$.

Soln:-

Given $f(x) = \dfrac{x^2+x-2}{x^3+6}$

W.K.T $\dfrac{u}{v} = \dfrac{vdu - udv}{v^2}$

$$f'(x) = \frac{(x^3+6)\, d/dx(x^2+x-2) - (x^2+x-2)\, d/dx(x^3+6)}{(x^3+6)^2}$$

$$= \frac{(x^3+6)(2x+1) - (x^2+x-2)(3x^2)}{(x^3+6)^2}$$

$$= \frac{2x^4+x^3+12x+6 - 3x^4-3x^3+6x^2}{(x^3+6)^2}$$

$$= \frac{-x^4-2x^3+6x^2+12x+6}{(x^3+6)^2}$$

Chain Rule :-

Eg - ① If $y = \sin(x^2)$, find $\frac{dy}{dx}$?

Soln: Given $y = \sin(x^2)$

$$\frac{dy}{dx} = \cos(x^2) \cdot (2x)$$

$$\frac{dy}{dx} = 2x \cos(x^2)$$

Eg - ② If $y = \cos(3x^2+4)$, find $dy/dx = ?$

Soln:-

Given $y = \cos(3x^2+4)$

$$dy/dx = \sin(3x^2+4)\ (6x)$$

$$dy/dx = 6x \sin(3x^2+4).$$

Eg - ③ If $y = (\sin x)^2$, find dy/dx

Soln Given $y = (\sin x)^2$

$$dy/dx = 2 \sin x \cdot \cos x.$$

Result:- If f is differentiable at 'a', then f is continuous at 'a'.

Proof:-

Given f is differentiable at 'a'

That is $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ Exists

To prove f is continuous at 'a'

ie, $\lim_{x \to a} f(x) = f(a)$

Consider, $f(x) - f(a) = \frac{f(x) - f(a)}{x - a} (x - a)$

$$\lim_{x \to a} f(x) - f(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \lim_{x \to a} (x - a)$$

$$= f'(a) \cdot (0)$$

$$= 0$$

$$\therefore \lim_{x \to a} f(x) = \lim_{x \to a} [f(a) + (f(x) - f(a))]$$

$$= \lim_{x \to a} f(a) + \lim_{x \to a} f(x) - f(a)$$

$$= \lim_{x \to a} f(a) + 0$$

$$= f(a)$$

$\therefore$ f is continuous at 'a'

CREATE

ILLUMINATE

INSPIRE

Derivative of Inverse Trigonometric function

Eg-① Differential Coefficient of $\tan^{-1}x$.

Soln

Given $y = \tan^{-1}x$.

ie $\tan y = x \longrightarrow ①$

Diff ①. w. r to. x.

$$\frac{d}{dx}(\tan y) = \frac{d}{dx}(x)$$

$$\sec^2 y \, \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

$$\therefore \frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$$

{By Eqn ①}

{W.K.T $\sec^2\theta = 1+\tan^2\theta$}

IIIly $\frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2}$

Eg-② Differential coefficient of $\sin^{-1}x$.

Soln

Given $y = \sin^{-1}x$

ie, $\sin y = x \longrightarrow ①$

Diff ①. w. r. to. x.

$$\cos y \left(\frac{dy}{dx}\right) = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

$$= \frac{1}{\sqrt{1-\sin^2 y}}$$

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$$

{W.K.T
$\sin^2\theta + \cos^2\theta = 1$
$\cos^2\theta = 1-\sin^2\theta$
$\cos\theta = \sqrt{1-\sin^2\theta}$
and Eqn ①

IIIly $\frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}$

Eg - (3) Find the derivative of $\left[\frac{\log x}{\sin x}\right]$.

Soln:

Given $y = \frac{\log x}{\sin x}$.

$\left\{ \text{W.K.T } \frac{u}{v} = \frac{v\,du - u\,dv}{v^2} \right\}$

Here, $u = \log x$ | $v = \sin x$

$du = 1/x$ | $dv = \cos x$

$$y' = \frac{dy}{dx} = \frac{\sin x(1/x) - \log x\,(\cos x)}{(\sin x)^2}$$

$$y' = \frac{1/x\,\sin x - \cos x\,\log x}{\sin^2 x}$$

Eg - (4) Find the derivative $f(x) = x e^x \sin x$.

Soln:

Given $f(x) = x e^x \sin x$.

$\{ \text{W.K.T } uvw = du\,vw + u\,dv\,w + uv\,dw \}$

$$f'(x) = (1)\,e^x \sin x + x\,(e^x) \sin x + x\,e^x (\cos x)$$

$$f'(x) = e^x \sin x + x e^x \sin x + x e^x \cos x$$ //

Eg-5 Find the derivative of $f(x) = \cos^{-1}\left[\frac{b+a\cos x}{a+b\cos x}\right]$.

Soln:

Let $f(x) = \cos^{-1}\left[\frac{b+a\cos x}{a+b\cos x}\right]$

Put $u = \frac{b+a\cos x}{a+b\cos x}$ → (1), Hence $f(x) = \cos^{-1}(u)$

Diff. using chain rule $f'(x) = \frac{-1}{\sqrt{1-u^2}} \frac{du}{dx}$. → (2)

and $u = \frac{b+a\cos x}{a+b\cos x}$

Diff. w.r.to. x

$$\frac{du}{dx} = \frac{(a+b\cos x)(-a\sin x) - (b+a\cos x)(-b\sin x)}{(a+b\cos x)^2}$$

$$= \frac{-a^2\sin x - ab\,\cancel{\sin x\cos x} + b^2\sin x + ab\,\cancel{\sin x\cos x}}{(a+b\cos x)^2}$$

$$= \frac{-a^2\sin x + b^2\sin x}{(a+b\cos x)^2}$$

$$\frac{du}{dx} = \frac{(b^2-a^2)\sin x}{(a+b\cos x)^2} \longrightarrow (3)$$

use (1) & (3) in Eqn (2)

$$(2) \Rightarrow f'(x) = \frac{-1}{\sqrt{1-u^2}} \frac{du}{dx}$$

$$= \frac{-1}{\sqrt{1-\left(\frac{b+a\cos x}{a+b\cos x}\right)^2}} \frac{(b^2-a^2)\sin x}{(a+b\cos x)^2}$$

$$= \frac{-1}{\sqrt{1-\frac{(b+a\cos x)^2}{(a+b\cos x)^2}}} \quad \frac{(b^2-a^2)\sin x}{(a+b\cos x)^2}$$

$$= \frac{-1}{\sqrt{\frac{(a+b\cos x)^2-(b+a\cos x)^2}{(a+b\cos x)^2}}} \times \frac{(b^2-a^2)\sin x}{(a+b\cos x)^2}$$

$$= \frac{-1}{\sqrt{\frac{(a^2+b^2\cos^2 x+2ab\cos x)-(b^2+a^2\cos^2 x+2ab\cos x)}{(a+b\cos x)^2}}} \times \frac{(b^2-a^2)\sin x}{(a+b\cos x)^2}$$

$$= \frac{-1}{\sqrt{\frac{a^2+b^2\cos^2 x+\cancel{2ab\cos x}-b^2-a^2\cos^2 x-\cancel{2ab\cos x}}{(a+b\cos x)^2}}} \times \frac{(b^2-a^2)\sin x}{(a+b\cos x)^2}$$

$$= \frac{-1}{\sqrt{\frac{a^2-a^2\cos^2 x-b^2-b^2\cos^2 x}{(a+b\cos x)^2}}} \times \frac{(b^2-a^2)\sin x}{(a+b\cos x)^2}$$

$$= \frac{-1}{\sqrt{\frac{(a^2-b^2)-\cos^2 x(a^2+b^2)}{(a+b\cos x)^2}}} \times \frac{(b^2-a^2)\sin x}{(a+b\cos x)^2}$$

$$= \frac{-1}{\sqrt{\frac{(a^2-b^2)(1-\cos^2 x)}{(a+b\cos x)^2}}} \times \frac{(b^2-a^2)\sin x}{(a+b\cos x)^2}$$

$$= \frac{-1}{\sqrt{\frac{(a^2-b^2)\sin^2 x}{(a+b\cos x)^2}}} \times \frac{(b^2-a^2)\sin x}{(a+b\cos x)^2}$$

$$= \frac{-1}{\dfrac{\sqrt{(a^2-b^2)\sin^2 x}}{(a+b\cos x)}} \times \frac{(b^2-a^2)\sin x}{(a+b\cos x)^2}$$

$$= \frac{-\cancel{(a+b\cos x)}}{\sqrt{(a^2-b^2)\sin^2 x}} \times \frac{(b^2-a^2)\sin x}{(a+b\cos x)^{\cancel{2}}}$$

$$= \frac{-(b^2-a^2)\sin x}{\sqrt{(a^2-b^2)\sin^2 x}\;(a+b\cos x)}$$

$$= \frac{(a^2-b^2)\sin x}{\sqrt{(a^2-b^2)\sin^2 x}\;(a+b\cos x)}$$

$$f'(x) = \frac{\sqrt{a^2-b^2}}{(a+b\cos x)}$$

Eg-6 Find the derivative $y = \log(\tan e^x)$.

Soln. Given $y = \log(\tan e^x)$

Put $\boxed{v = e^x} \longrightarrow (1)$

ie $y = \log(\tan v)$

and Put $\boxed{u = \tan v} \longrightarrow (2)$

ie $y = \log u \longrightarrow (3)$

① $\Rightarrow$ $v = e^x$

$$\frac{dv}{dx} = e^x$$

② $\Rightarrow$ $u = \tan v$

$$\frac{du}{dx} = \sec^2 v \cdot \frac{dv}{dx}$$

$$= \sec^2 v \cdot e^x$$

$$\frac{du}{dx} = \sec^2(e^x)(e^x)$$

③ $\Rightarrow$ $y = \log v.$

Diff. w. r to. x

$$y' = \frac{1}{u}\,\frac{du}{dx}$$

$$= \frac{1}{\tan v} \cdot \sec^2(e^x)(e^x)$$

$$= \frac{1}{\tan(e^x)}(e^x)\sec^2(e^x)$$

$$\left\{ \begin{array}{c} \text{w.k.t} \\ \frac{1}{\tan\theta} = \cot\theta \end{array} \right\}$$

$$= \cot(e^x)(e^x)\sec^2(e^x)$$

$$y' = e^x \cot(e^x)\sec^2(e^x)$$ //

Eg-(8) Find the derivative of $\tanh^{-1}(\sin x)$.

Soln:-

Given $f(x) = \tanh^{-1}(\sin x)$

Let $\boxed{u = \sin x} \longrightarrow (1)$

$$\frac{du}{dx} = \cos x \longrightarrow (2)$$

$\therefore f(x) = \tanh^{-1}(u)$

Diff. w.r.to. x

$$f'(x) = \frac{1}{1-u^2} \, du/dx$$

$$= \frac{1}{1-\sin^2 x} \cos x$$

$$= \frac{1}{\cos^2 x} \cos x$$

$$= \frac{1}{\cos x}$$

$$f'(x) = \sec x$$

Eg - 7 Find the derivative of $\tanh^{-1}(\tan x/2)$.

Sol^n:

Given $f(x) = \tanh^{-1}(\tan x/2)$

Let $\boxed{u = \tan(x/2)} \longrightarrow (1)$

$$\frac{du}{dx} = \frac{1}{2}\sec^2(x/2) \longrightarrow (2)$$

ii $f(x) = \tanh^{-1}(u) \longrightarrow (3)$

Diff. w.r.to. 'x'

$$f'(x) = \frac{1}{1-u^2}\, du/dx$$

use (1) & (2)

$$= \frac{1}{1-\tan^2(x/2)}\ \frac{1}{2}\sec^2(x/2)$$

$$= \frac{\frac{1}{2}\sec^2(x/2)}{1-\tan^2(x/2)} = \frac{\frac{1}{2}\sec^2(x/2)}{1-\frac{\sin^2(x/2)}{\cos^2(x/2)}}$$

$$= \frac{\frac{1}{2}\sec^2(x/2)}{\left(\frac{\cos^2(x/2)-\sin^2(x/2)}{\cos^2(x/2)}\right)} = \frac{\frac{1}{2}\frac{1}{\cos^2(x/2)}}{\cos^2(x/2)-\sin^2(x/2)} \times \cos^2(x/2)$$

$$= \frac{\frac{1}{2}\frac{\cancel{\cos^2(x/2)}}{\cancel{\cos^2(x/2)}}}{\cos^2(x/2)-\sin^2(x/2)} = \frac{1}{2}\,\frac{1}{\cos^2 x/2 - \sin^2 x/2}$$

$$f'(x) = \frac{1}{2}\sec x$$

Example. Find $\frac{dy}{dx}$, if $y = x^2 e^{2x} (x^2+1)^4$

Soln:

Given that $y = x^2 e^{2x} (x^2+1)^4$

$\{$ w.k.t $uvw = du\, vw + u\, dv\, w + uv\, dw \}$

$$\frac{dy}{dx} = 2x\, e^{2x} (x^2+1)^4 + x^2\, 2e^{2x} (x^2+1)^4 + x^2 e^{2x}\, 4(x^2+1)^3 (2x)$$

$$= 2x\, e^{2x} (x^2+1)^4 + 2x^2 e^{2x} (x^2+1)^4 + 8x^3 e^{2x} (x^2+1)^3$$

Example. Find y' for $\cos(xy) = 1 + \sin y$

Soln:-

Given that

$$\cos(xy) = 1 + \sin y$$

$$-\sin(xy)[xy' + y(1)] = \cos y\, (y')$$

$$-\sin(xy)[xy' + y] = y' \cos y$$

$$[xy' + y] = \frac{y' \cos y}{-\sin(xy)}$$

$$\frac{xy' + y}{y'} = \frac{-\cos y}{\sin(xy)}$$

$$\frac{x\cancel{y'}}{\cancel{y'}} + \frac{y}{y'} = \frac{-\cos y}{\sin(xy)}$$

$$x + \frac{y}{y'} = \frac{-\cos y}{\sin(xy)}$$

$$\frac{y}{y'} = \frac{-\cos y}{\sin(xy)} - x$$

$$\frac{y}{y'} = \frac{-\cos y}{\sin(xy)} - x$$

$$\frac{y}{y'} = \frac{-\cos y - x\sin(xy)}{\sin(xy)}$$

$$\frac{y'}{y} = \frac{\sin(xy)}{-\cos y - x\sin(xy)}$$

$$y' = \frac{y\sin(xy)}{-\cos y - x\sin(xy)}$$

CHAPTER - ~~1~~ 4 [MAXIMUM AND MINIMUM]

Defn:

Let c be a number in the domain D of a function f. the f(c) is the

(a) Absolute maximum value of f on D, if $f(c) \geq f(x)$, $\forall$ x in D.

(b) Absolute minimum value of f on D, if $f(c) \leq f(x)$, $\forall$ x in D.

Defn:

The number f(c) is a

(a) Local maximum value of f, if $f(c) \geq f(x)$, when x is near c.

(b) Local minimum value of f, if $f(c) \leq f(x)$, when x is near c.

Fermat's theorem:-

If f has local maximum (or) minimum at 'c' and if $f'(c)$ exists, then $f'(c) = 0$.

Eg: ① Find the critical number of $f(x) = x^{3/5}(4-x)$

Soln:

Given $f(x) = x^{3/5}(4-x)$

$= 4x^{3/5} - x^{3/5}x^{1}$

$f(x) = 4x^{3/5} - x^{8/5}$

(iv) $f(x) = 4x^{3/5} - x^{8/5}$

$$f'(x) = 4 \cdot 3/5\, x^{3/5 - 1} - 8/5\, x^{8/5 - 1}$$

$$f'(x) = \frac{12}{5} x^{-2/5} - 8/5\, x^{3/5}$$

$$f'(x) = 0 \Rightarrow \frac{12}{5} x^{-2/5} - 8/5\, x^{3/5} = 0$$

$$\frac{12}{5} \frac{1}{x^{2/5}} - 8/5\, x^{3/5} = 0$$

(multiply by $5x^{2/5}$)

$$\frac{12}{\cancel{5}} \frac{1}{\cancel{x^{2/5}}} \cancel{5x^{2/5}} - 8/\cancel{5}\, x^{3/5} \cancel{5} x^{2/5} = 0$$

$$12 - 8x = 0$$

$$8x = 12$$

$$\boxed{x = 12/8} \Rightarrow \boxed{x = 3/2}$$

Eg - ②: Find the absolute maximum and minimum values of the function $f(x) = x^3 - 3x^2 + 1$, $-1/2 \le x \le 4$.

Soln.

Given $f(x) = x^3 - 3x^2 + 1$, if $-1/2 \le x \le 4$.

$$f'(x) = 3x^2 - 6x$$

put $f'(x) = 0 \Rightarrow 3x^2 - 6x = 0$

$$x^2 - 2x = 0$$

$$x(x-2) = 0$$

$\boxed{x = 0}$ | $x - 2 = 0$

$\boxed{x = 2}$

The Points are $x = 0, 2$

$\therefore$ $\boxed{f(0) = 1}$, $f(2) = (2)^3 - 3(2^2) + 1$

$= 8 - 3(4) + 1$

$= 8 - 12 + 1$

$\boxed{f(2) = -3}$

The values of f at the End Points of the interval are

$f(-\frac{1}{2}) = (-\frac{1}{2})^3 - 3(-\frac{1}{2})^2 + 1$

$= -\frac{1}{8} - \frac{3}{4} + 1 = \frac{-1-6+8}{8} = \frac{1}{8}$

$\boxed{f(-\frac{1}{2}) = \frac{1}{8}}$

$f(4) = (4)^3 - 3(4^2) + 1$

$= 64 - 3(16) + 1 = 64 - 48 + 1 = 17$

$\boxed{f(4) = 17}$

Comparing these 4-values

The absolute maximum is $f(4) = 17$

The absolute minimum is $f(2) = -3$

Eg-3 Find the absolute minimum and maximum values for the function $f(x) = x - 2\sin x$, $0 \leq x \leq 2\pi$.

Soln: Given $f(x) = x - 2\sin x$, $0 \leq x \leq 2\pi$

$f'(x) = 1 - 2\cos x$

Put $f'(x)=0 \Rightarrow 1-2\cos x=0$

$$2\cos x=1$$

$$\cos x=1/2$$

$$x=\cos^{-1}(1/2)$$

$$\therefore x=\pi/3 \text{ (or) } x=5\pi/3$$

$f(\pi/3) = \pi/2 - 2\sin(\pi/3) = \pi/2 - \cancel{2}\sqrt{3}/\cancel{2}$

$= \pi/2 - \sqrt{3} = -0.684853$

$f(5\pi/3) = \frac{5\pi}{3} - 2\sin(5\pi/3) = \frac{5\pi}{3} + \sqrt{3} = 6.968039$

The values of f at the End Points are.

$f(0) = 0 - 2\sin(0) = 0$

$f(2\pi) = 2\pi - 2\sin(2\pi) = 2\pi - (0) = 2\pi = 6.28$

Comparing the four values.

The absolute minimum value is $f(\pi/3) = \pi/3 - \sqrt{3}$

the absolute maximum value is $f(5\pi/3) = \frac{5\pi}{3} + \sqrt{3}$.

ROLLE'S THEOREM:-

Let 'f' be a function that satisfies the following three assumptions.

i) f is continuous on the closed interval $[a,b]$

ii) f is differentiable on the open interval (a,b)

iii) $f(a) = f(b)$, then there is a number 'C' in (a,b) s.t $f'(c) = 0$.

Eg-① Verify the function $f(x) = 5 - 12x + 3x^2$ in $[1,3]$ satisfies the Rolle's theorem.

Soln:

Given that $f(x) = 5 - 12x + 3x^2$

Since, $f(x)$ is Polynomial and hence its continuous and differentiable.

$$\therefore f(1) = 5 - 12(1) + 3(1)^2 = 5 - 12 + 3 = -4$$

$$f(3) = 5 - 12(3) + 3(3)^2 = 5 - 36 + 27 = -4$$

$$\therefore f(1) = f(3)$$

$f(x)$ satisfied the Rolle's theorem

$$\therefore f'(x) = -12 + 6x$$

$$f'(c) = 0 \Rightarrow -12 + 6c = 0$$

$$6c = 12$$

$$c = 12/6$$

$$\boxed{c = 2}$$

Eg - 2 $\quad f(x) = \cos 2x$ in $[\pi/8, 7\pi/8]$.

Soln: Given that $f(x) = \cos 2x$ in $[\pi/8, 7\pi/8]$

Since, $f(x)$ is continuous and differentiable in $[\pi/8, 7\pi/8]$.

is $f(\pi/8) = \cos(2\pi/8) = \cos \pi/4 = \frac{1}{\sqrt{2}}$

$f(7\pi/8) = \cos 2(7\pi/8) = \cos \frac{7\pi}{4} = \frac{1}{\sqrt{2}}$

$\therefore \quad f(\pi/8) = f(7\pi/8)$

$\therefore$ $f(x)$ satisfies the Rolle's theorem.

$f'(x) = -2\sin 2x$

$\therefore f'(c) = 0 \Rightarrow -2\sin 2c = 0$

$\sin 2c = 0$

$2c = n\pi$

$\boxed{c = n\pi/2}$

Put $n=1 \Rightarrow c = \pi/2$.

∴ MEAN VALUE THEOREM :-

let f be a function that satisfies the two ~~three~~ assumptions.

i) f is continuous on the closed interval $[a,b]$

ii) f is differentiable on the open interval (a,b)

then there is a number c in (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

Eg:- ① verify the mean value theorem on the given interval, and find all values of c in that interval that satisfy the theorem (a) $f(x) = x^2 - x$ in $[-3,5]$

(b) $f(x) = \sqrt{25-x^2}$ in $[-5,3]$.

Soln:-

(a) Given that $f(x) = x^2 - x$ in $[-3,5]$

Since, $f(x) = x^2 - x$ is a Polynomial. It is Continuous on $[-3,5]$ and differentiable on $(-3,5)$.

By mean value theorem, $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

$$f(a) = f(-3) = (-3)^2 - (-3) = 9+3 = 12$$

$$f(b) = f(5) = (5)^2 - 5 = 25 - 5 = 20$$

$$\therefore \frac{f(b)-f(a)}{b-a} = \frac{f(5)-f(-3)}{5-(-3)} = \frac{20-12}{5+3}$$

$$= 8/8 \boxed{=1.}$$

Then $f'(x) = 2x-1$

$\therefore f'(c) = \cancel{0} \Rightarrow 2c \cancel{-1} = \cancel{0}$

$2c = 1+1 \Rightarrow 2c = 2$

$\cancel{c = 1/2}$ $c = 2/2$ $\boxed{c=1}$

$\therefore c = 1$ ~~(f(x) = f(c) = 1/2)~~ is in $(-3, 5)$

(b) Given that $f(x) = \sqrt{25-x^2}$ in $[-5, 3]$

Since $f(x)$ is continuous on $[-5, 3]$ and differentiable in $[-5, 3]$.

By M.V.T $f'(c) = \dfrac{f(b)-f(a)}{b-a}$.

$f(a) = f(-5) = \sqrt{25-(-5)^2} = \sqrt{25-25} = 0$

$f(b) = f(3) = \sqrt{25-(3)^2} = \sqrt{25-9} = \sqrt{16} = 4$

$\therefore \dfrac{f(b)-f(a)}{b-a} = \dfrac{4-0}{3-(-5)} = \dfrac{4}{3+5} = 4/8 = 1/2.$

then $f'(x) = \dfrac{-\cancel{2}x}{\cancel{2}\sqrt{25-x^2}} = \dfrac{-x}{\sqrt{25-x^2}}$.

$f'(c) = \dfrac{-c}{\sqrt{25-c^2}} = 1/2$

$-c = 1/2 \sqrt{25-c^2}$

$(-c)^2 = (1/2)^2 \left(\sqrt{25-c^2}\right)^2$

$$c^2 = 1/4\,(25 - c^2)$$

$$c^2 = \frac{25}{4} - \frac{c^2}{4}$$

$$c^2 + c^2/4 = \frac{25}{4}$$

$$\frac{4c^2 + c^2}{\cancel{4}} = \frac{25}{\cancel{4}}$$

$$5c^2 = 25$$

$$c^2 = 25/5 = 5$$

$$c^2 = 5$$

$\boxed{c = \sqrt{5}}$ which in in $(-5, 3)$

Eg - (3) For the function $f(x) = 3x^4 - 4x^3 - 12x^2 - 5$, where is it increasing and decreasing.

Soln:

Given that $f(x) = 3x^4 - 4x^3 - 12x^2 - 5$

$$\therefore f'(x) = 12x^3 - 12x^2 - 24x.$$

$$= 12[x^3 - x^2 - 2x] = 12x[x^2 - x - 2]$$

$$f'(x) = 12x(x-2)(x+1)$$

To apply increasing (or) decreasing test, we have to find where $f'(x) > 0$ and $f'(x) < 0$.

It depends on the singn of the three factory of $f'(x)$ namely, $12x$, $(x-2)$ and $(x+1)$.

The critical points are: 0, 2, and −1.

Interval	$12x$	$(x-2)$	$(x+1)$	$f'(x)$	$f(x)$
$x < -1$	−	−	−	−	decreasing on $(-\infty, -1)$
$-1 < x < 0$	−	−	+	+	Increasing on $(-1, 0)$
$0 < x < 2$	+	−	+	−	decreasing on $(0, 2)$
$x > 2$	+	+	+	+	Increasing on $(2, \infty)$

Eg-③

Find the local maximum and minimum values of the function $f(x) = x + 2\sin x$, $0 \leq x \leq 2\pi$.

Soln:

Given that $f(x) = x + 2\sin x$. $[0, 2\pi]$

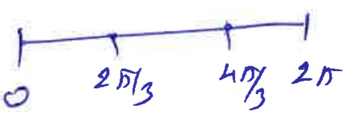
then $f'(x) = 1 + 2\cos x$

The critical points are $f'(x) = 0 \Rightarrow 1 + 2\cos x = 0$

$2\cos x = -1$

$\cos x = -1/2$

The solution are $x = 2\pi/3$ and $\frac{4\pi}{3}$



Interval	$f'(x) = 1 + 2\cos x$	$f(x)$
$0 < x < 2\pi/3$	+	Increasing on $(0, 2\pi/3)$
$2\pi/3 < x < 4\pi/3$	−	Decreasing on $(2\pi/3, 4\pi/3)$
$4\pi/3 < x < 2\pi$	+	Increasing on $(4\pi/3, 2\pi)$

Since, $f'(x)$ changes from +ve to −ve at $2\pi/3$
hence $f(x)$ attains local maximum at $2\pi/3$

∴ The maximum value is $f(2\pi/3) = 2\pi/3 + 2\sin(2\pi/3)$

$$= 2\pi/3 + \cancel{2}\left(\sqrt{3}/\cancel{2}\right)$$
$$= 2\pi/3 + \sqrt{3}$$
$$= 3.83$$

Since, $f'(x)$ changes from −ve to +ve at $4\pi/3$
hence $f(x)$ attains local minimum at $4\pi/3$

∴ The minimum value is $f(4\pi/3) = \frac{4\pi}{3} + 2\sin(4\pi/3)$

$$= \frac{4\pi}{3} + \cancel{2}(-\sqrt{3}/\cancel{2})$$
$$= \frac{4\pi}{3} - \sqrt{3}$$
$$= 2.46$$

Example-④ For the function $f(x) = 2x^3 + 3x^2 - 36x$.

(i) Find the intervals on which it is increasing or decreasing.

(ii) Find the local maximum and minimum values of f.

(iii) Find the intervals of concavity and the inflection points.

Soln:

(i) Given $f(x) = 2x^3 + 3x^2 - 36x$

$$f'(x) = 6x^2 + 6x - 36$$
$$= 6(x^2 + x - 6)$$

$$\begin{array}{c|c} & -6 \\ \hline 3 & -2 \\ \hline & 1 \end{array}$$

$$f'(x) = 6(x+3)(x-2)$$

To apply Increasing or decreasing test, we have to find where $f'(x) > 0$ and $f'(x) < 0$.

It depends on the sign of the two factors $(x+3)(x-2)$.

Interval	$(x+3)$	$(x-2)$	$f'(x)$	$f(x)$
$x < -3$	-ve	-ve	+ve	Increasing on $(-\infty, -3)$
$-3 < x < 2$	+ve	-ve	-ve	Decreasing on $(-3, 2)$
$x > 2$	+ve	+ve	+ve	Increasing on $(2, \infty)$

(ii) Since $f'(x)$ changes from +ve to -ve at $x = -3$

hence $f(x)$ attains local maximum at $x = -3$

$\therefore$ The Local maximum is $f(x) = 2x^3 + 3x^2 - 36x$

$$f(-3) = 2(-3)^3 + 3(-3)^2 - 36(-3)$$
$$= 2(-27) + 3(9) + 108$$
$$= -54 + 27 + 108$$
$$f(-3) = 81$$

Since $f'(x)$ changes from -ve to +ve at $x = 2$.

hence $f(x)$ attains local minimum at $x = 2$

$\therefore$ The Local minimum value is

$$f(2) = 2(2)^3 + 3(2)^2 - 36(2)$$
$$= 2(8) + 3(4) - 72$$
$$= 16 + 12 - 72$$
$$f(2) = -44$$

(iii) Let $f'(x) = 6x^2+6x-36$ and $f''(x) = 12x+6$

$$f''(x) = 0 \Rightarrow 12x+6=0$$
$$12x = -6$$
$$x = -6/12 \Rightarrow \boxed{x = -1/2}$$

$f''(x) > 0, \Rightarrow x > -1/2$ and $f''(x) < 0 \Rightarrow x < -1/2$

Thus f is concave upward on $(-1/2, \infty)$

The inflection point at $[-1/2, f(-1/2)]$

$$f(x) = 2(x^3) + 3x^2 - 36x$$
$$f(-1/2) = 2(-1/2)^3 + 3(-1/2)^2 - 36(-1/2)$$
$$= \cancel{2}(-1/\cancel{8}_4) + 3(1/4) + \cancel{36}^{18}(1/\cancel{2})$$
$$= -1/4 + 3/4 + 18 = 2/4 + 18$$
$$= 1/2 + 18 = \frac{1+36}{2}$$
$$\boxed{f(-1/2) = 37/2}$$

The inflection point at $[-1/2, 37/2]$

Example-(5) For the function $f(x) = 2+2x^2-x^4$, find the intervals of increase or decrease, local maximum and minimum values, the intervals of concavity and the inflection points.

Soln:-

Let $f(x) = 2+2x^2-x^4$

$$f'(x) = 4x - 4x^3$$
$$= 4x(1-x^2)$$
$$f'(x) = 4x(1+x)(1-x)$$

{ W.K.T $a^2-b^2 = (a+b)(a-b)$

To apply Increasing or decreasing test, we have to find where $f'(x) > 0$ and $f'(x) < 0$.

Interval	$4x$	$(1-x)$	$(1+x)$	$f'(x)$	$f(x)$
$x < -1$	$-ve$	$+ve$	$-ve$	$+ve$	Increasing on $(-\infty, -1)$
$-1 < x < 0$	$-ve$	$+ve$	$+ve$	$-ve$	decreasing on $(-1, 0)$
$0 < x < 1$	$+ve$	$+ve$	$+ve$	$+ve$	Increasing on $(0, 1)$
$x > 2$	$+ve$	$-ve$	$+ve$	$-ve$	decreasing on $(1, \infty)$

(ii) $f(x)$ changes from $+ve$ to $-ve$ at $\boxed{x=-1}$

hence $f(x)$ attains local maximum at $x=-1$

$\therefore$ Local maximum is $f(x) = 2+2x^2-x^4$

$$f(-1) = 2+2(-1)^2-(-1)^2$$
$$= 2+2-1 = 4-1$$
$$\boxed{f(-1) = 3}$$

$f(x)$ changes from $-ve$ to $+ve$ at $\boxed{x=0}$

hence $f(x)$ attains local minimum at $x=0$

$\therefore$ Local minimum value is $f(0) = 2+0-0$

$$\boxed{f(0) = 2}$$

$f(x)$ changes from $+ve$ to $-ve$ at $\boxed{x=1}$

hence $f(x)$ attains local maximum at $\boxed{x=1}$

$\therefore$ Local maximum value is

$$f(1) = 2+2(1)^2-(1)^4$$
$$= 2+2-1$$
$$= 4-1$$
$$\boxed{f(1) = 3}$$

Example - 6

Find the local maximum and minimum values of $f(x) = \sqrt{x} - \sqrt[4]{x}$ using both first and second derivative test.

Soln:

Given that $f(x) = \sqrt{x} - \sqrt[4]{x}$

ie, $f(x) = x^{1/2} - x^{1/4}$

$$f'(x) = \tfrac{1}{2} x^{-1/2} - \tfrac{1}{4} x^{-3/4}$$

$$= \tfrac{1}{4} x^{-3/4} \left[2x^{1/4} - 1 \right]$$

$$f'(x) = \frac{2\sqrt[4]{x} - 1}{4\sqrt[4]{x^3}}$$

(i) First derivative test :- $f'(x) = 0$

Let $2\sqrt[4]{x} - 1 > 0 \Rightarrow 2\sqrt[4]{x} \geq 1$

$$\sqrt[4]{x} \geq \tfrac{1}{2}$$

$$(\sqrt[4]{x})^4 \geq (\tfrac{1}{2})^4$$

$$x \geq \tfrac{1}{16}.$$

$f'(x) > 0$ when $x > 1/16$

Similarly. $f'(x) < 0 \Rightarrow 0 < x < 1/16$

Since $f'(x)$ changes from negative to positive at $x = 1/16$.

The Local minimum value is

$$f(x) = \sqrt{x} - \sqrt[4]{x}$$

$$f(1/16) = \sqrt{1/16} - \sqrt[4]{1/16} = 1/4 - 1/2 = -1/4$$

$$\boxed{f(1/16) = -1/4}$$

(ii) <u>Second derivative test</u> :-

Let $f'(x) = \frac{1}{2} x^{-1/2} - \frac{1}{4} x^{-3/4}$

$$f''(x) = -\frac{1}{4} x^{-3/2} + \frac{3}{16} x^{-7/4}.$$

$$f''(x) = -\frac{1}{4} \frac{1}{\sqrt{x^3}} + \frac{3}{16} \frac{1}{\sqrt[4]{x^7}}$$

$$f'(x) = 0 \Rightarrow 2\sqrt[4]{x} - 1 = 0 \Rightarrow \boxed{x = 1/16}$$

$$f''(x) = \frac{-1}{4\sqrt{x^3}} + \frac{3}{16\sqrt[4]{x^7}}$$

$$f''(1/16) = \frac{-1}{4\sqrt{(1/16)^3}} + \frac{3}{16\sqrt[4]{(1/16)^7}}$$

$$= \frac{-1}{4\left(\frac{1}{64}\right)} + \frac{3}{16\left(\frac{1}{128}\right)} \neq \text{th}$$

$$= \frac{-1}{(1/16)} + \frac{3}{(1/8)} = -16 + 3(8)$$

$$= -16 + 24$$

$$f''(1/16) = 8 > 0$$

The Local minimum value ~~at~~ $x = 1/16$ is

$$f(x) = \sqrt{x} - \sqrt[4]{x}$$

$$f(1/16) = \sqrt{1/16} - \sqrt[4]{1/16}$$

$$= 1/4 - 1/2$$

$$\boxed{f(1/16) = -1/4}$$